Application of Geometry Expressions to Theorems from Machine Proofs in Geometry

Saltire Software internal Report 2016-1

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In this document we apply Geometry Expressions to a set of theorems from [1]. While not human readable, Geometry Expressions does provide a form of proof. The objective here is to calibrate its performance against this given set of examples.

1 Introduction

The goal in this document is to try and "prove" the theorem using Geometry Expressions. If we cannot, we try to use an external CAS to help. If this fails, we will look at special cases we can try and address. If all fail we will look to improve Geometry Expressions!

Examples are numbered as they appear in the book.

The bulk of this document consists of screenshots taken from Geometry Expressions. These screenshots indicate both the model, which would be input by the user, and the result, constituting the Geometry Expressions proof, which was generated automatically.

Examples 6.1 to 6.207 are treated.

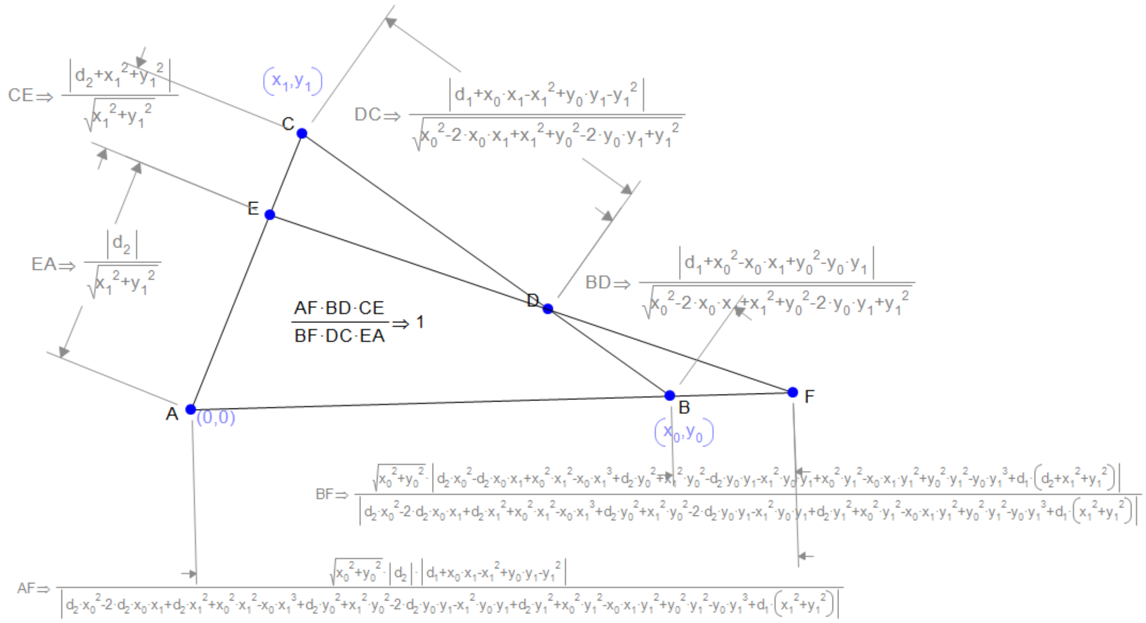
1.1 Results

Example by example comments are given below. In summary, however, of the 207 examples, 196 are solved in a reasonable amount of time on a modern PC by Geometry Expressions, using an initial formulation.

5 examples failed on the initial formulation, but succeeded on a second formulation (6.69, 6.122, 6.149, 6.163, 6.182). In 3 cases (6.22, 6.104, 6.200), the internal Geometry Expressions CAS was unable to simplify the result, but Maple was able to complete the simplification. In 2 cases, the initial formulation failed, but succeeded after supplying a simple intermediate result (6.172, 6.174). In 1 case (6.109) neither Geometry Expressions' CAS nor Maple is able to simplify the result. In 1 case (6.128) the result in the book is incorrect.

2 Geometry of Incidence

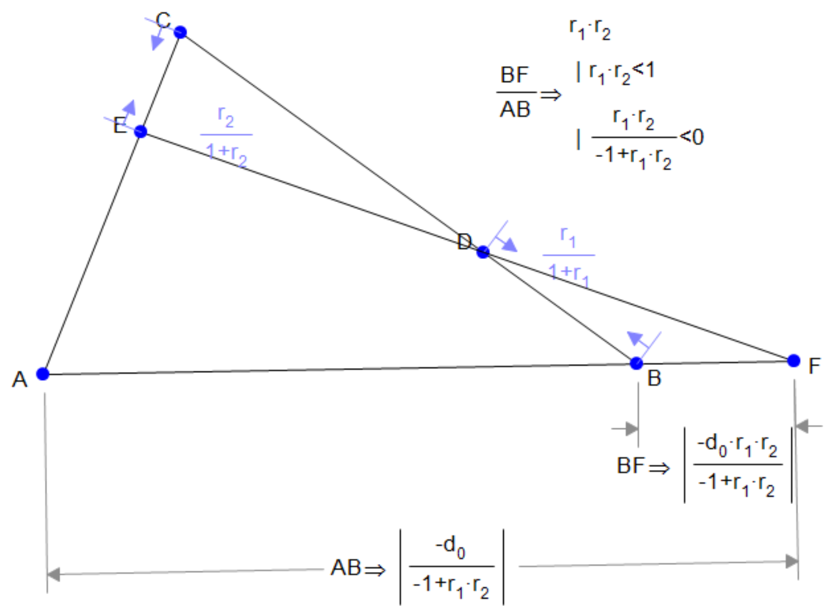
2.1 Menelaus Theorem



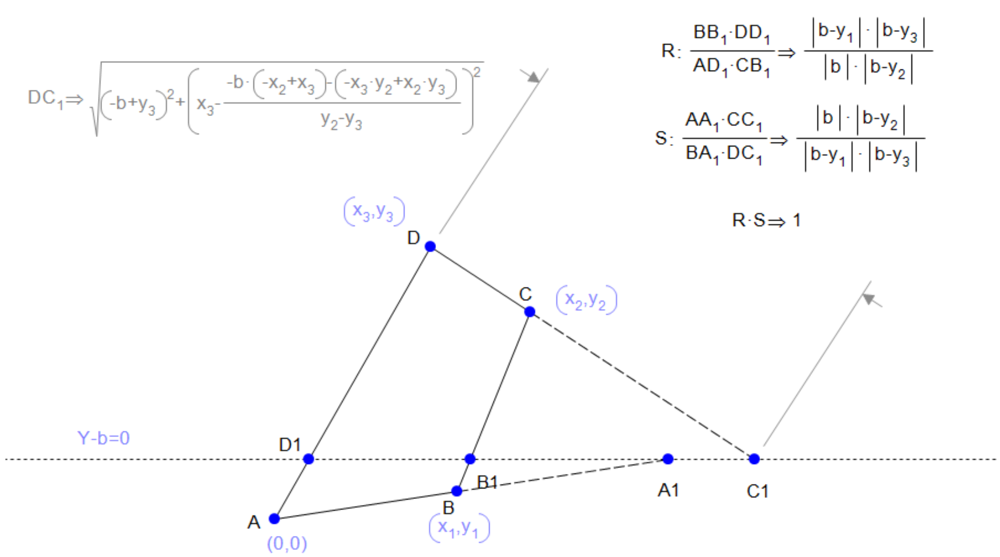
Without loss of generality, we make A be the origin. AF and BF have been hidden as they are a bit large.

6.1 (Converse of Menelaus Theorem)

Ratio r_1 is defined to be BD/DC. GX point proportional constraint accepts the ratio BD/BC. So we enter $r_1/(1+ r_1)$.

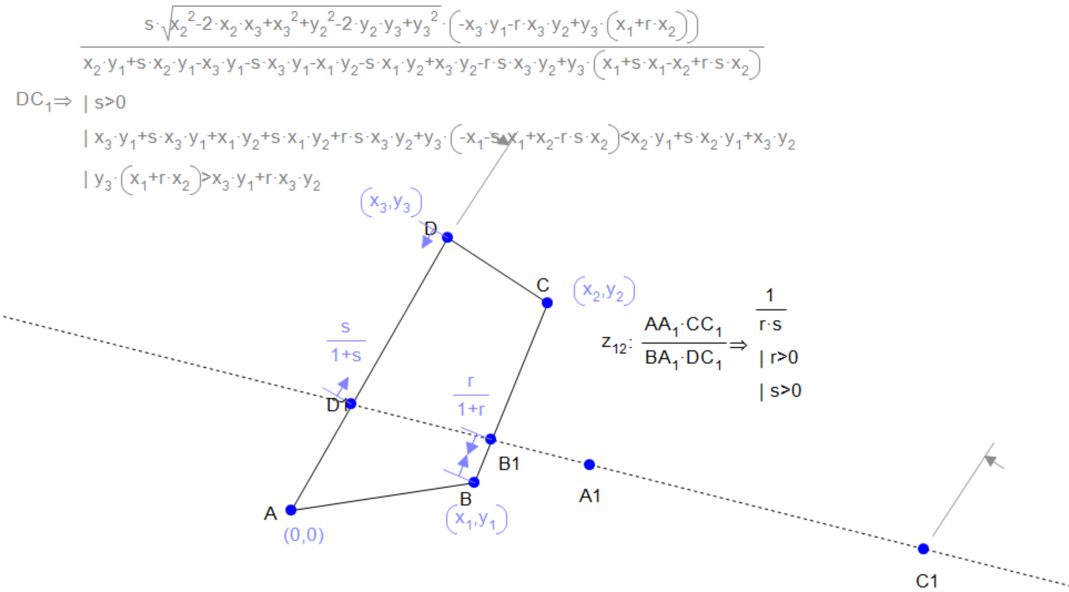


6.2 Menelaus Theorem for a Quadrilateral



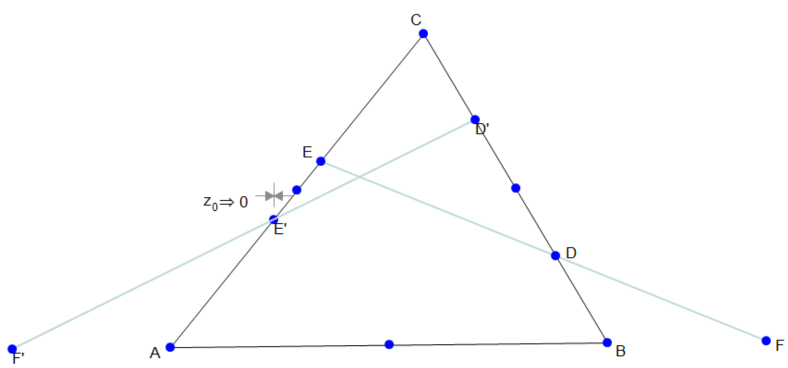
Here we have elected without loss of generality to make A the origin and the line y=b.

6.3 Menelaus Theorem for a Quadrilateral (done a little differently)



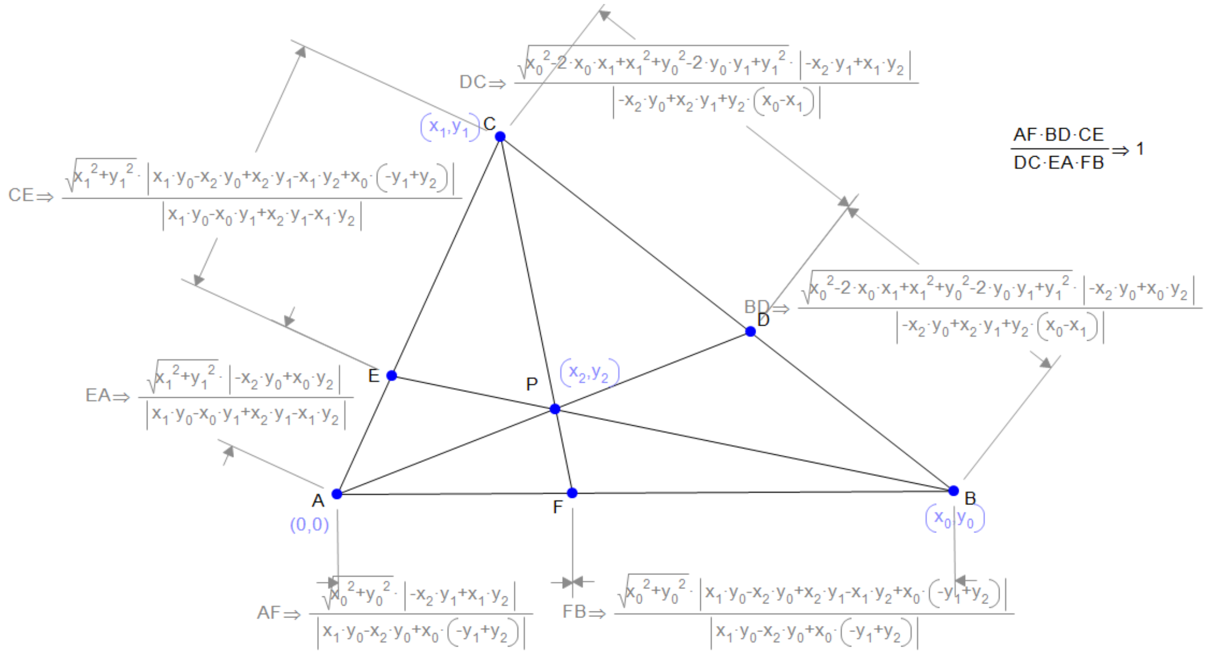
In this one, we specify the coordinates of the quad points and proportions for B1 and D1, and measure the other proportions. A useful trick is to make one of the coordinates (0,0), which cuts down on the complexity of the results.

6.4 The isotomic points of three collinear points are collinear



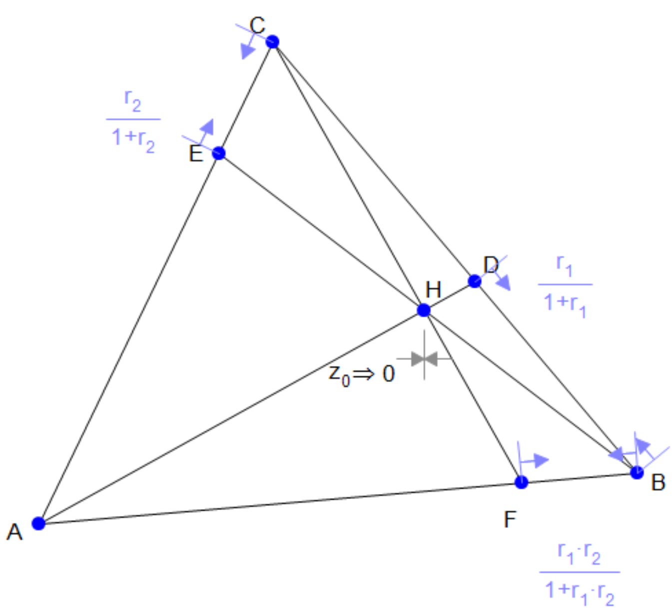
Isotomic points are created by reflecting points in segment midpoints. (reflection in a point is done in Geometry Expressions using dilation of scale -1).

2.2 Ceva's Theorem



Here is a direct proof of Ceva's theorem. In fact, here the individual steps of an analytical geometry proof can be seen sketched.

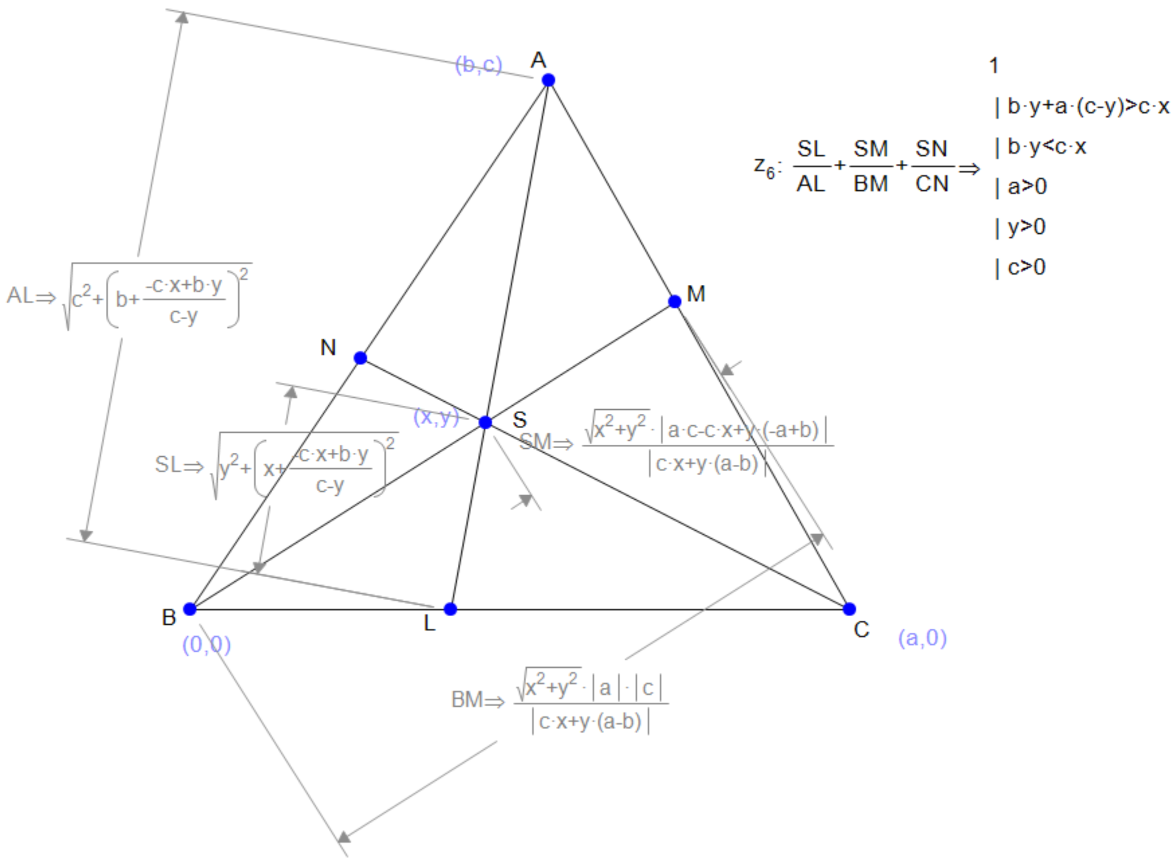
6.5 Converse of Ceva's Theorem



Here we have set H to be the intersection of lines AD and BE. We measure its distance to line CF

6.6 Ratios on the Cevian

If LMN is the Cevian Triangle of point S in triangle ABC, we have $\frac{SL}{AL} + \frac{SM}{BM} + \frac{SN}{CN} = 1$



SN and CN are hidden for clarity.

6.7 Sub triangle areas

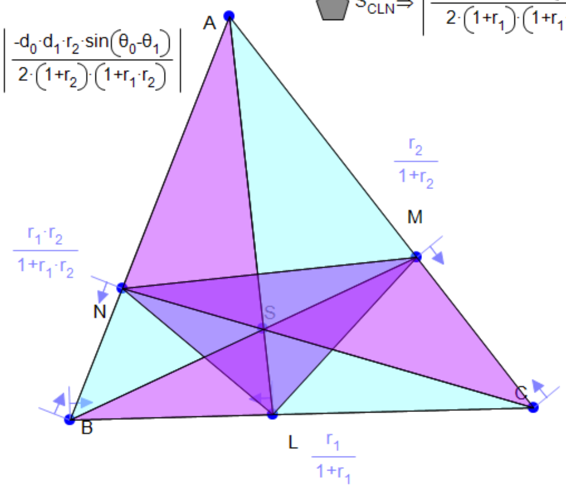
If LMN is the Cevian Triangle of point S in triangle ABC, we have $\frac{S_{AML} \cdot S_{BNM} \cdot S_{CLN}}{S_{ANL} \cdot S_{BLM} \cdot S_{CNM}}$

$$S_{ALN} \Rightarrow \left| \frac{-d_0 \cdot d_1 \cdot r_1 \cdot \sin(\theta_0-\theta_1)}{2 \cdot (1+r_1) \cdot (1+r_1 \cdot r_2)} \right| \qquad S_{AML} \Rightarrow \left| \frac{-d_0 \cdot d_1 \cdot \sin(\theta_0-\theta_1)}{2 \cdot (1+r_1) \cdot (1+r_2)} \right|$$

$$S_{BML} \Rightarrow \left| \frac{-d_0 \cdot d_1 \cdot r_1 \cdot r_2 \cdot \sin(\theta_0-\theta_1)}{2 \cdot (1+r_1) \cdot (1+r_2)} \right| \qquad S_{BNM} \Rightarrow \left| \frac{-d_0 \cdot d_1 \cdot r_1 \cdot r_2 \cdot \sin(\theta_0-\theta_1)}{2 \cdot (1+r_2) \cdot (1+r_1 \cdot r_2)} \right|$$

$$S_{CLN} \Rightarrow \left| \frac{-d_0 \cdot d_1 \cdot r_1 \cdot r_2 \cdot \sin(\theta_0-\theta_1)}{2 \cdot (1+r_1) \cdot (1+r_1 \cdot r_2)} \right|$$

$$S_{CNM} \Rightarrow \left| \frac{-d_0 \cdot d_1 \cdot r_2 \cdot \sin(\theta_0-\theta_1)}{2 \cdot (1+r_2) \cdot (1+r_1 \cdot r_2)} \right|$$



$$\frac{S_{AML} \cdot S_{BNM} \cdot S_{CLN}}{S_{ALN} \cdot S_{BML} \cdot S_{CNM}} \Rightarrow \begin{array}{l} 1 \\ \mid \frac{\sin(\theta_0-\theta_1)}{(1+r_1) \cdot (1+r_2)} < 0 \\ \mid \frac{r_1 \cdot r_2 \cdot \sin(\theta_0-\theta_1)}{(1+r_1) \cdot (1+r_2)} < 0 \\ \mid \frac{r_1 \cdot \sin(\theta_0-\theta_1)}{(1+r_1) \cdot (1+r_1 \cdot r_2)} < 0 \\ \mid \frac{r_1 \cdot r_2 \cdot \sin(\theta_0-\theta_1)}{(1+r_1) \cdot (1+r_1 \cdot r_2)} < 0 \\ \mid \frac{r_2 \cdot \sin(\theta_0-\theta_1)}{(1+r_2) \cdot (1+r_1 \cdot r_2)} < 0 \\ \mid \frac{r_1 \cdot r_2 \cdot \sin(\theta_0-\theta_1)}{(1+r_2) \cdot (1+r_1 \cdot r_2)} < 0 \end{array}$$

In the above diagram, we have left the coordinates out but specified the Cevian using ratios. Below we do the same thing but leaving out the ratios and constraining all three Cevians to pass through S. In this diagram we do put in coordinates for A B and C, setting BC to lie along the x axis.

$$S_{ALN} \Rightarrow \left| \frac{-c \cdot (c \cdot x - b \cdot y)}{2 \cdot (-c+y)} + \frac{a \cdot c \cdot y \cdot (c \cdot x - b \cdot y)}{2 \cdot (-c+y) \cdot (b \cdot y - c \cdot (-a+x))} \right|$$

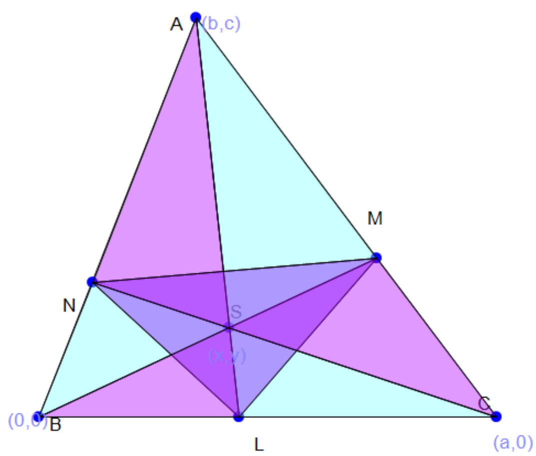
$$S_{AML} \Rightarrow \left| \frac{c \cdot \left(-a \cdot c^2 \cdot x + c^2 \cdot x^2 + y^2 \cdot \left(-a \cdot b + b^2\right) + y \cdot (a \cdot b \cdot c + a \cdot c \cdot x - 2 \cdot b \cdot c \cdot x)\right)}{2 \cdot (c-y) \cdot (-c \cdot x + y \cdot (-a+b))} \right|$$

$$S_{BML} \Rightarrow \left| \frac{a \cdot c \cdot y \cdot (c \cdot x - b \cdot y)}{2 \cdot (-c+y) \cdot (-c \cdot x + y \cdot (-a+b))} \right|$$

$$S_{BNM} \Rightarrow \left| \frac{a^2 \cdot c \cdot \left(-c \cdot x \cdot y + b \cdot y^2\right)}{2 \cdot (a \cdot c - c \cdot x + b \cdot y) \cdot (-c \cdot x + y \cdot (-a+b))} \right|$$

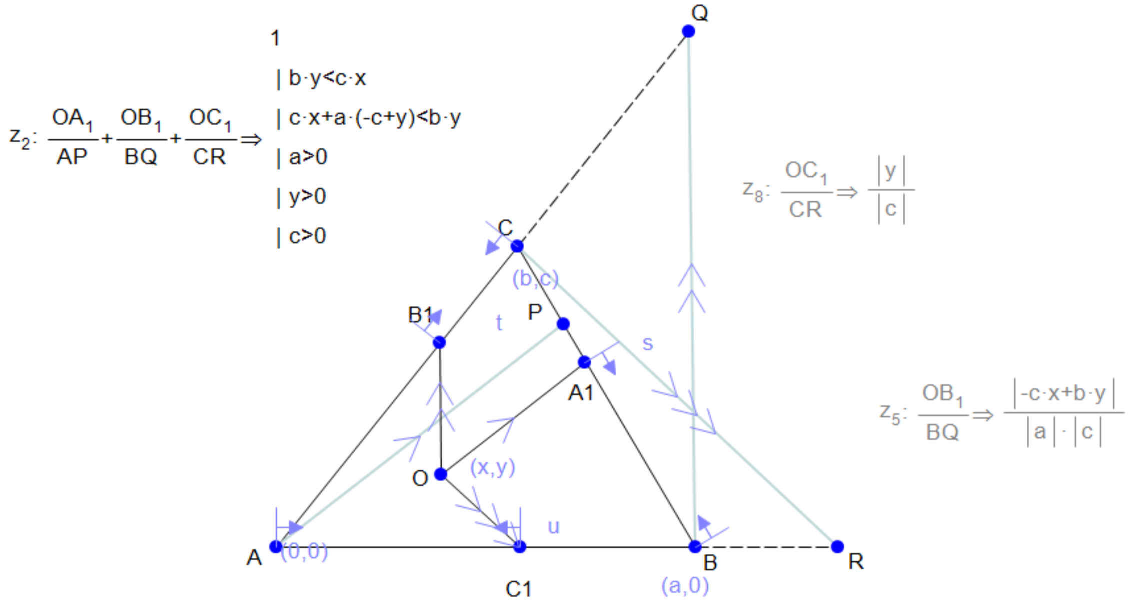
$$S_{CNM} \Rightarrow \left| \frac{a^2 \cdot c \cdot \left(y^2 \cdot (a-b) + y \cdot (-a \cdot c + c \cdot x)\right)}{2 \cdot (a \cdot c - c \cdot x + b \cdot y) \cdot (-c \cdot x + y \cdot (-a+b))} \right|$$

$$S_{CLN} \Rightarrow \left| \frac{a \cdot c \cdot \left(y^2 \cdot (a-b) + y \cdot (-a \cdot c + c \cdot x)\right)}{2 \cdot (-c+y) \cdot (a \cdot c - c \cdot x + b \cdot y)} \right|$$



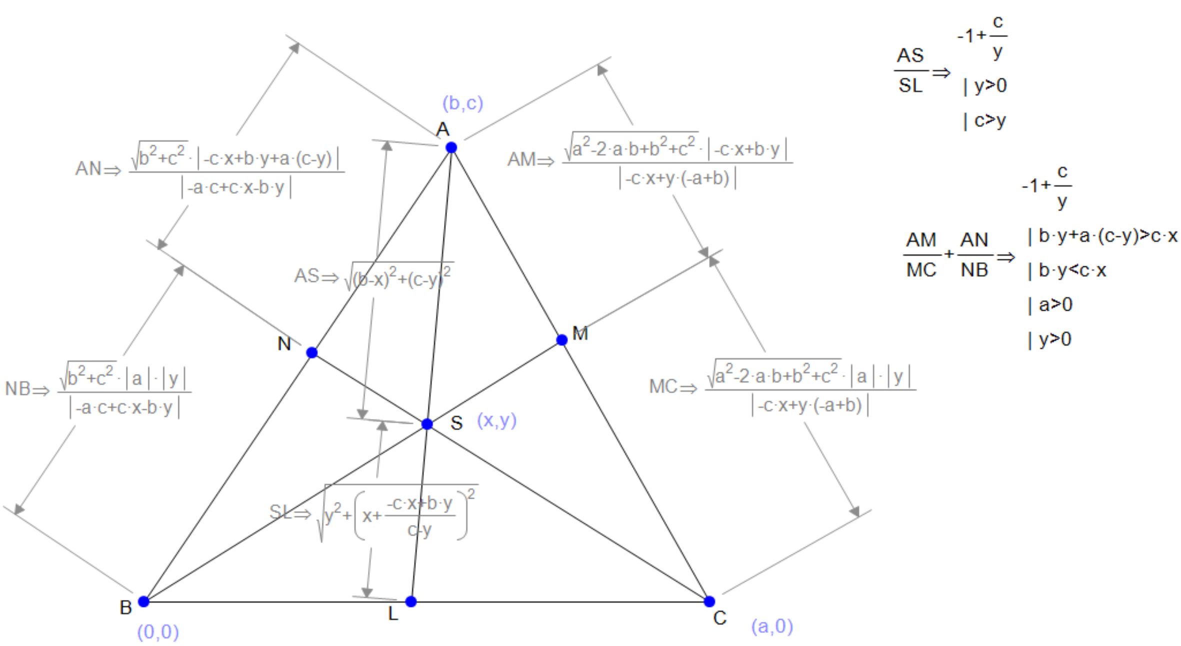
$$\frac{S_{AML} \cdot S_{BNM} \cdot S_{CLN}}{S_{ALN} \cdot S_{BML} \cdot S_{CNM}} \Rightarrow \begin{array}{l} 1 \\ \mid \frac{c \cdot \left(-a \cdot c^2 \cdot x + c^2 \cdot x^2 + y^2 \cdot \left(-a \cdot b + b^2\right) + y \cdot (a \cdot b \cdot c + a \cdot c \cdot x - 2 \cdot b \cdot c \cdot x)\right)}{(c-y) \cdot (-c \cdot x + y \cdot (-a+b))} > 0 \\ \mid \frac{c \cdot \left(a \cdot c^2 \cdot x - c^2 \cdot x^2 + y^2 \cdot \left(a \cdot b - b^2\right) + y \cdot (-a \cdot b \cdot c - a \cdot c \cdot x + 2 \cdot b \cdot c \cdot x)\right)}{(c-y) \cdot (a \cdot c - c \cdot x + b \cdot y)} > 0 \\ \mid \frac{a \cdot c \cdot \left(-c \cdot x \cdot y + b \cdot y^2\right)}{(c-y) \cdot (-c \cdot x + y \cdot (-a+b))} > 0 \\ \mid \frac{c \cdot \left(-c \cdot x \cdot y + b \cdot y^2\right)}{(a \cdot c - c \cdot x + b \cdot y) \cdot (-c \cdot x + y \cdot (-a+b))} > 0 \\ \mid \frac{c \cdot \left(y^2 \cdot (a-b) + y \cdot (-a \cdot c + c \cdot x)\right)}{(a \cdot c - c \cdot x + b \cdot y) \cdot (-c \cdot x + y \cdot (-a+b))} > 0 \\ \mid \frac{a \cdot c \cdot \left(y^2 \cdot (-a+b) + y \cdot (a \cdot c - c \cdot x)\right)}{(c-y) \cdot (a \cdot c - c \cdot x + b \cdot y)} > 0 \end{array}$$

6.8 Ratios of parallels

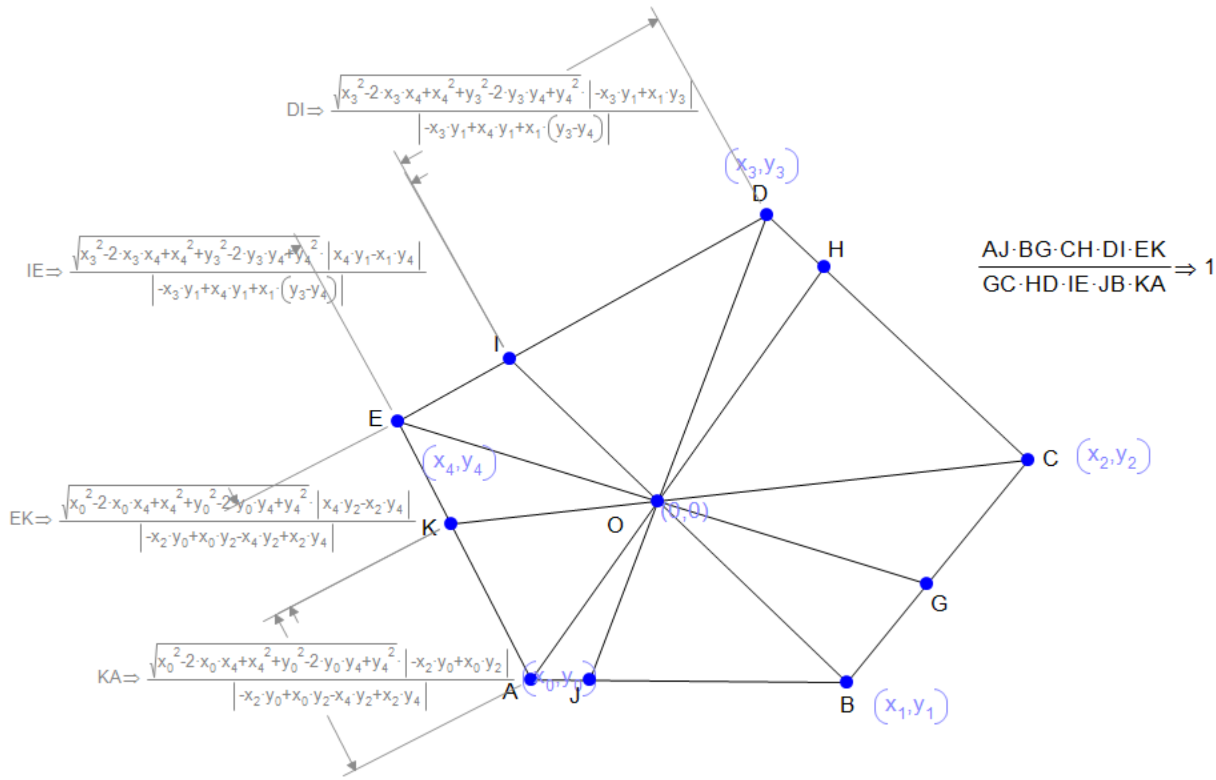


6.9 More ratios

If LMN is the Cevian triangle of the point S for the triangle ABC, we have: $\frac{AS}{SL} = \frac{AM}{MC} + \frac{AN}{NB}$



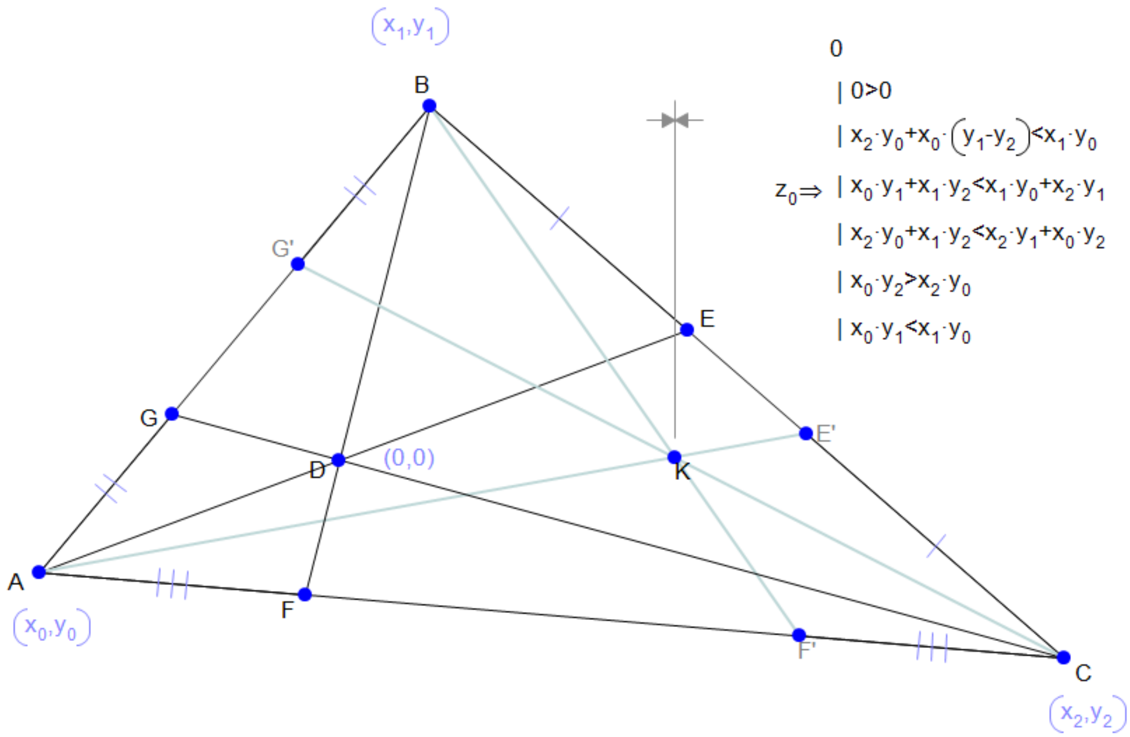
6.10 Ceva's Theorem for a pentagon



For simplicity, we put point O at the origin. We have hidden 6 similar intermediate results for clarity.

6.11 Isotomics of Cevian point

If the three lines joining three points marked on the sides of a triangle to the opposite vertices are concurrent, the same is true of the isotomics of the given points.

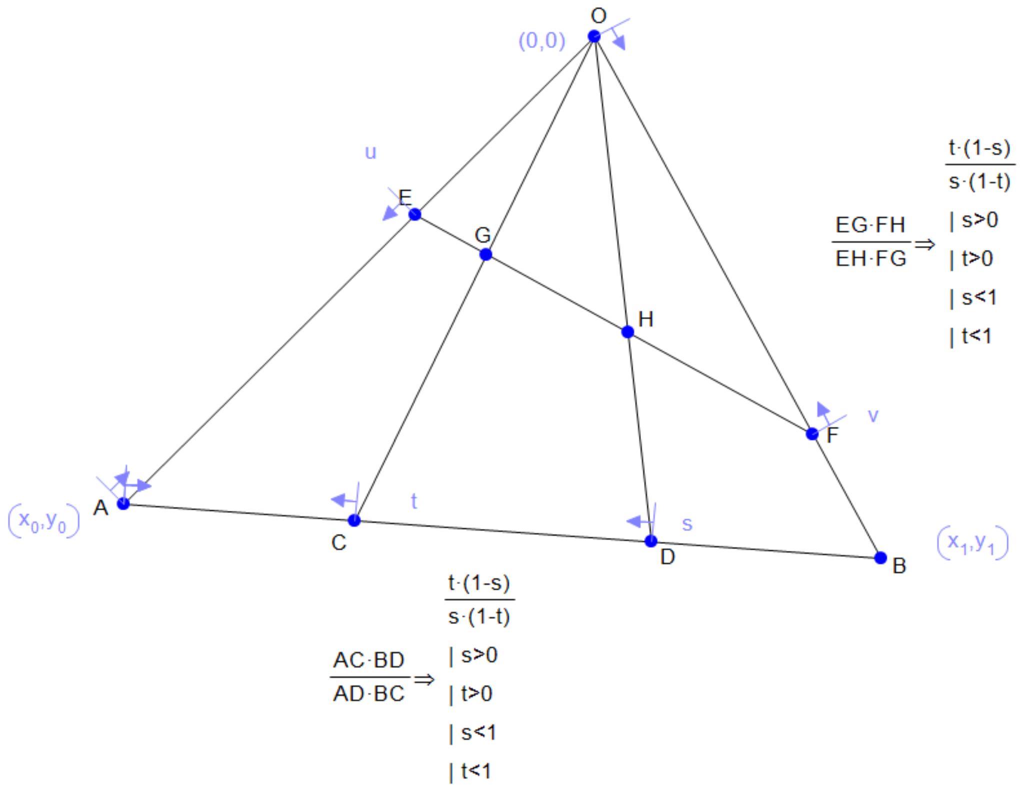


K is defined to be the intersection of AE' and BF' we measure its distance to CG'

2.3 The Cross Ratio and Harmonic Points

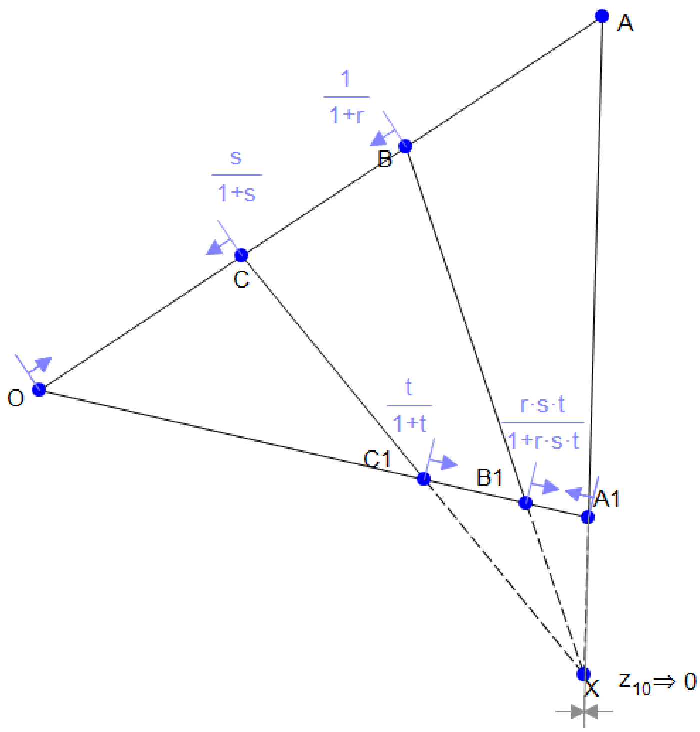
Let A,B,C,D be four collinear points. The cross ratio denoted (ABCD) = $\frac{CA/_{CB}}{DA/_{DB}}$

6.12 The cross ratio of four points on a line is unchanged by projection



6.13

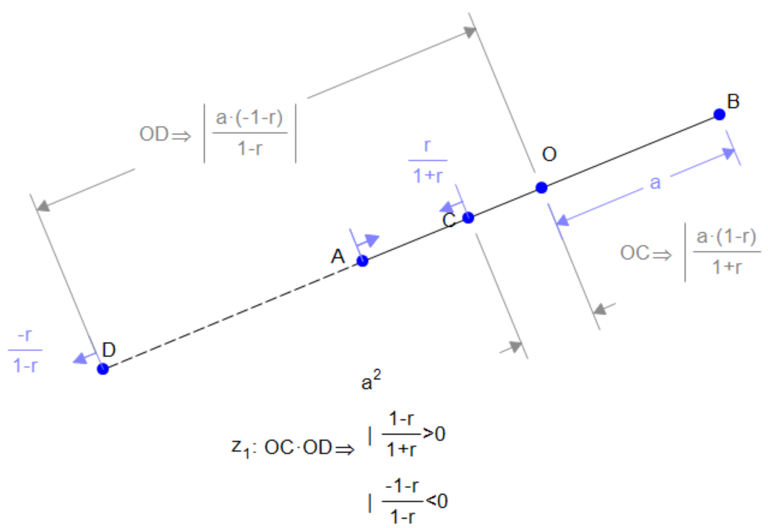
Two lines $OABC$ and $OA_1B_1C_1$ intersect at point O. If $(OABC) = (OA_1B_1C_1)$ then AA_1, BB_1, CC_1 are concurrent.



Point proportional constraints are rigged so that the cross ratio relationship holds. H is the intersection of CC1 and BB1. We measure its distance to line AA1.

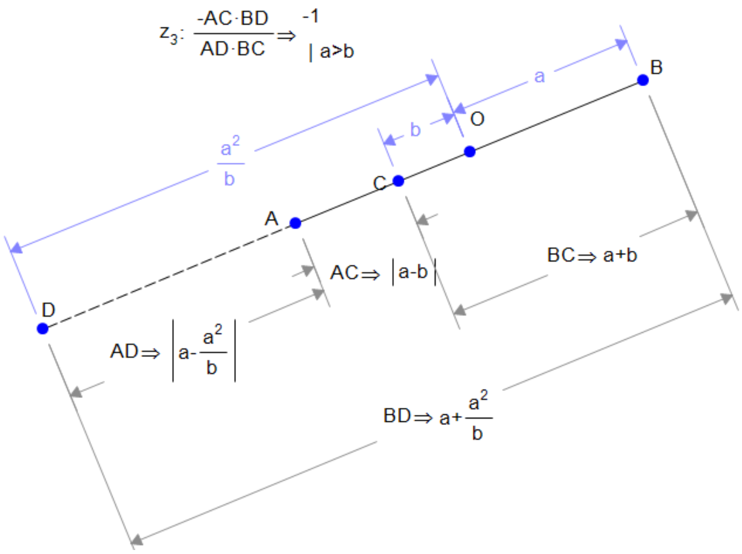
6.14

Let A,B,C,D be four harmonic points and O the midpoint of AB. Then $OC \cdot OD = OA^2$



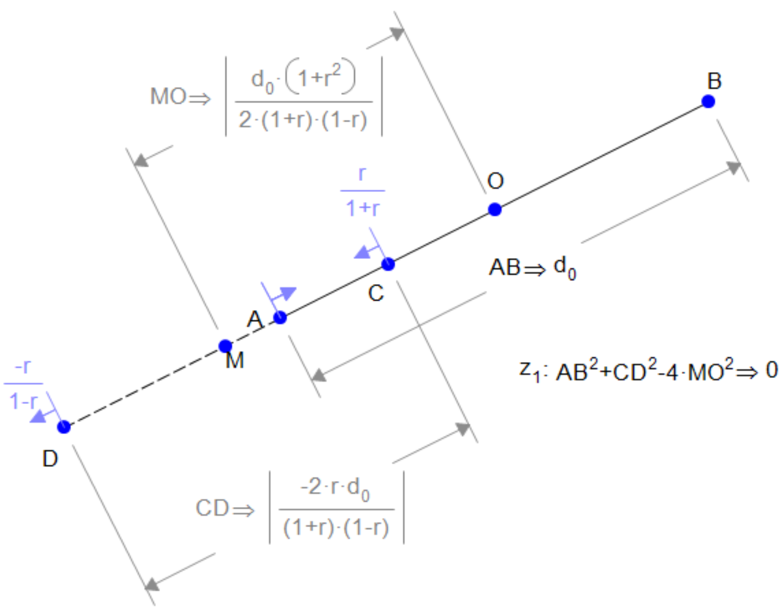
6.15 Converse of 6.14

If O is the midpoint of AB and if C,D are points on the line such that $OC \cdot OD = OA^2$ then A,B,C,D are a harmonic sequence.



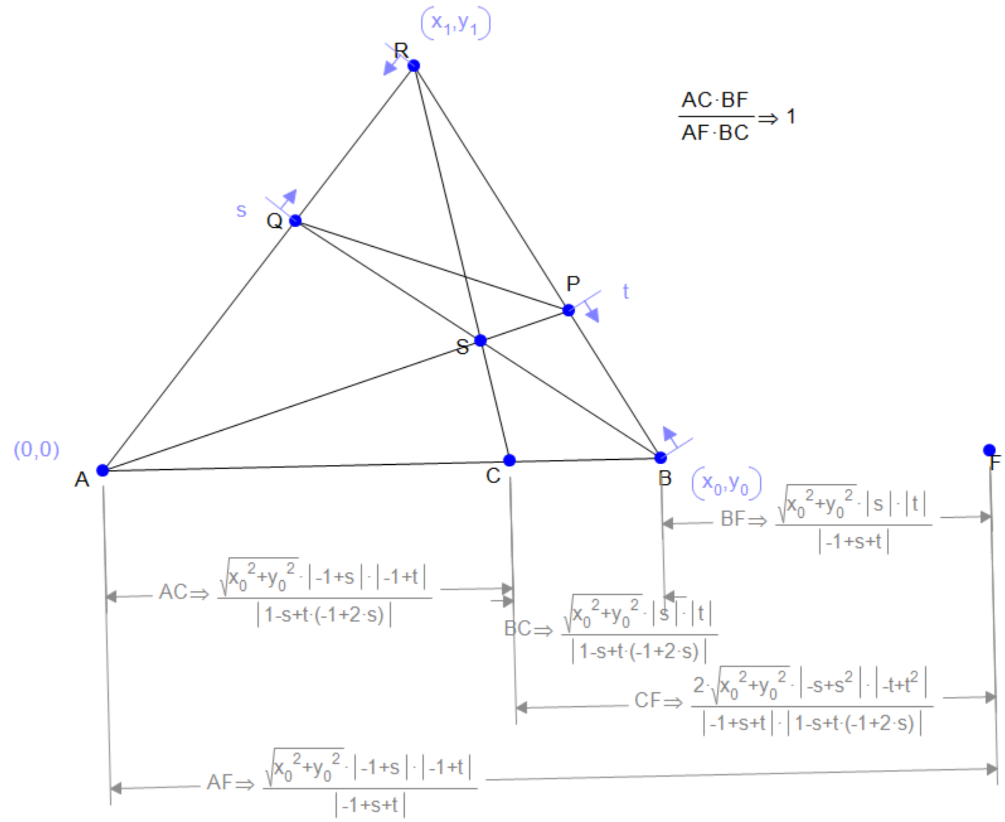
6.16

The sum of the squares of two harmonic segments is equal to four times the square of the distance between the midpoints of the segments.



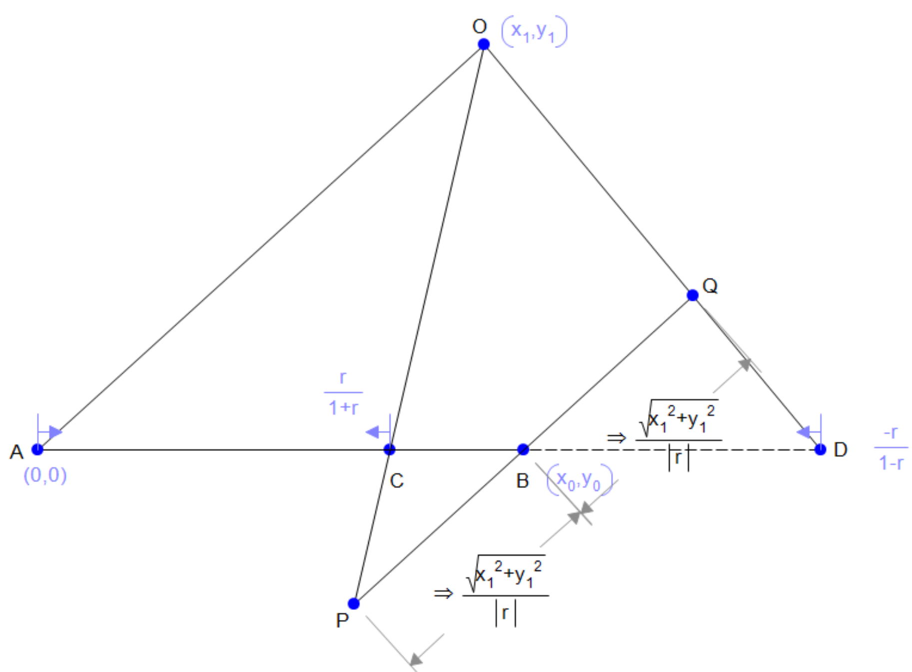
6.17

Let A,B,R be three points on a plane, Q and P be points on line AR and BR respectively. S is the intersection of QB and AP. C is the intersection of RS and AB. F is the intersection of QP and AB. Show that (ABCF) = -1.



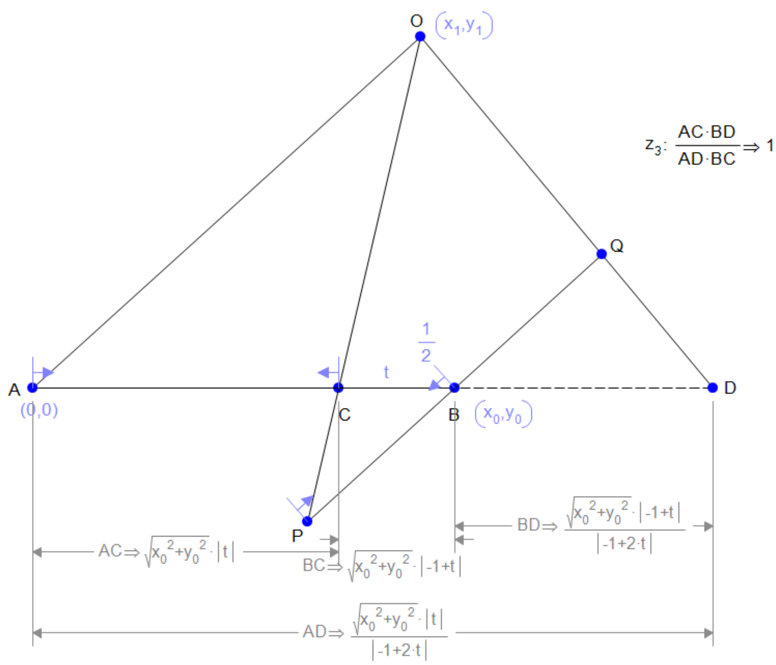
6.18

Given (ABCD)=-1 and a point O outside the line AB; if a parallel through B to OA meets OC,OD in P,Q, we then have PB=BQ.



6.19

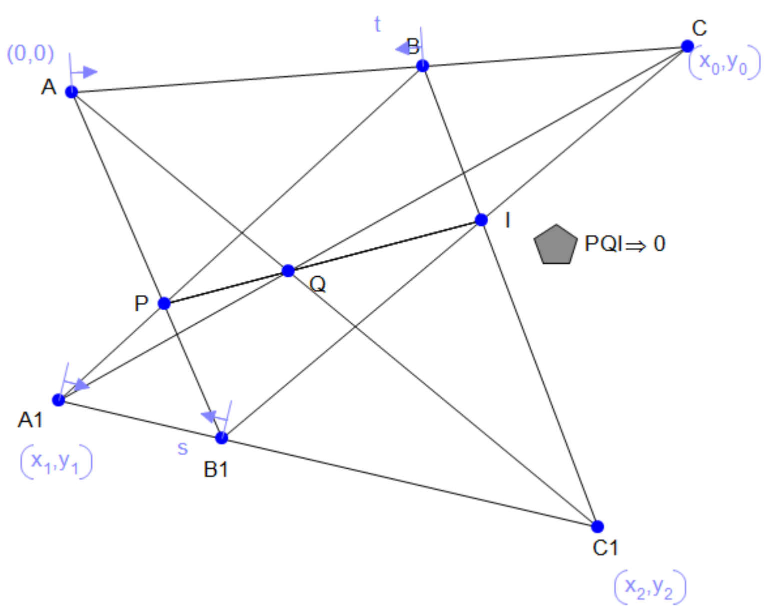
The converse of 6.18



2.4 Pappus Theorem and Desargues Theorem

6.20 Pappus Theorem

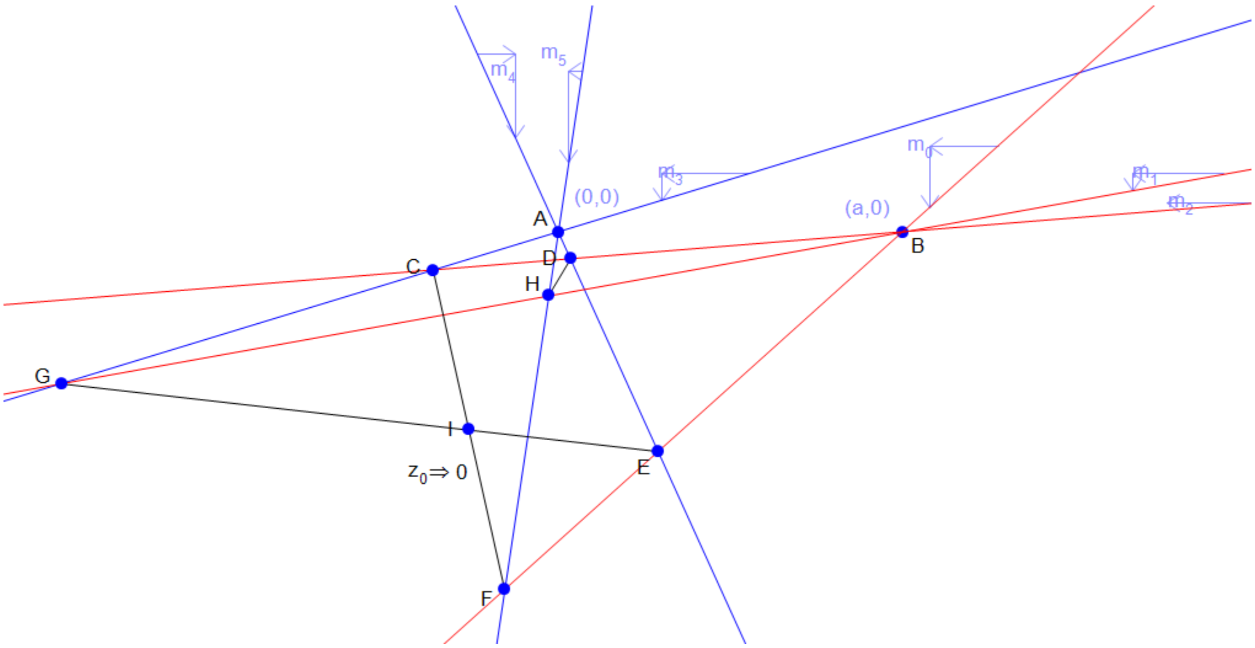
Let ABC and $A_1B_1C_1$ be two lines. Let P be the intersection of AB_1 and A_1B. Let Q be the intersection of AC_1 and A_1C. Let S be the intersection of BC_1 and B_1C. P, Q and S are collinear.



We could look for the distance between Q and the line PI. Geometry Expressions will return 0 for this but takes a few minutes. Faster is to show that the area of triangle PQI is 0.

6.21 Dual of Pappus

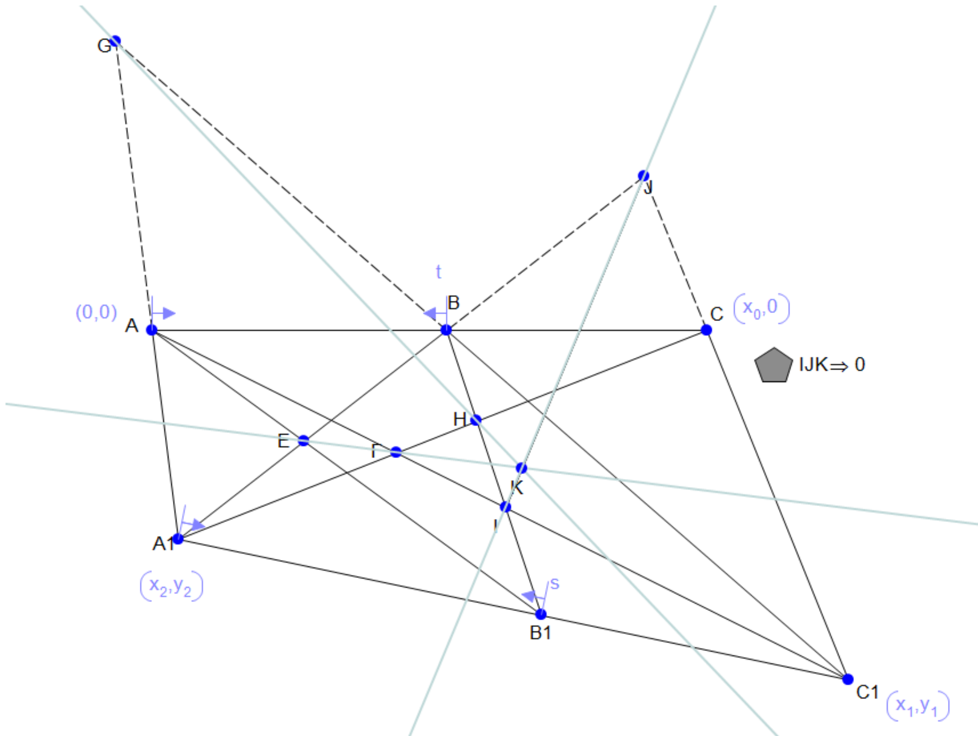
CD, EF and GH are intersections in pairs between 3 concurrent lines through A and B. Show that CF, GE and DH are concurrent.



We have constrained the locations of A and B and the slopes of all the lines. I is the intersection of GE and CF. We output the perpendicular distance from I to line DH.

6.22 Triples of Pappus Lines are concurrent

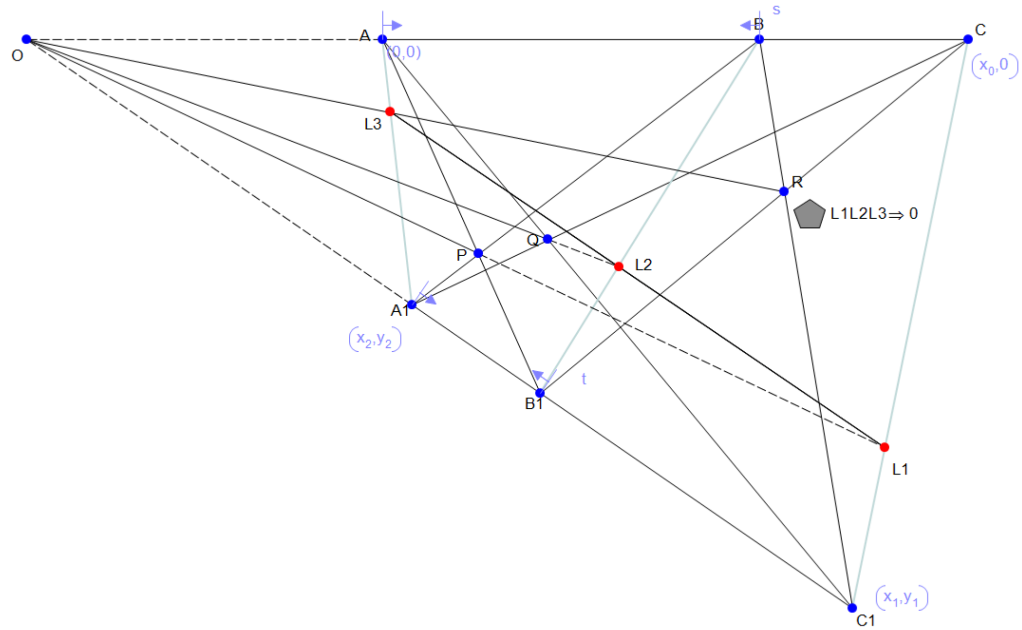
In the Pappus configuration we create a line L_1 joining the intersection of A_1B and AB_1 and the intersection of CA_1 and AC_1; line L_2 joining the intersection of BC_1 and AA_1 and the intersection of BB_1 and CA_1; line L_3 joining the intersection of BB_1 and AC_1 and the intersection of BA_1 and CC_1. Lines L_1, L_2, L_3 are concurrent.



In the model, we show the area of IJK is zero. An initial attempt had C at generic location. This produced a rational for the area which took a long time to simplify in Geometry Expressions (30 minutes or so), but which simplified in under a second in Maple.

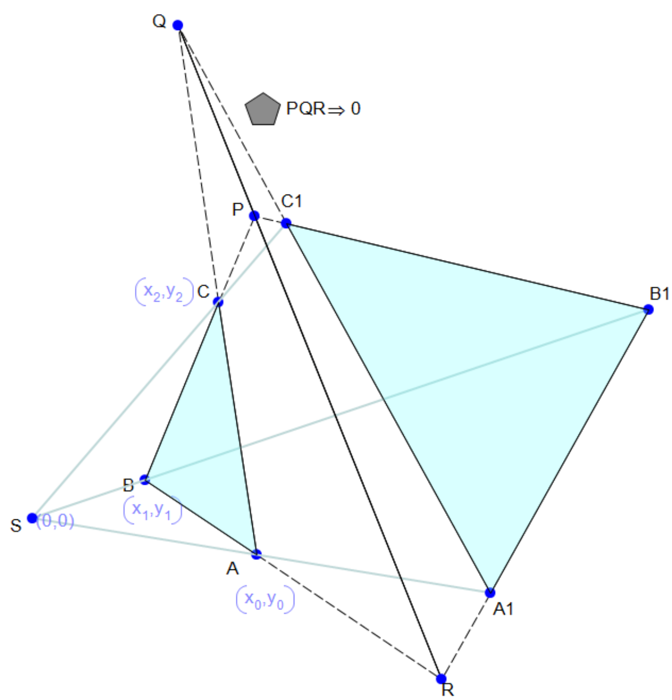
6.23 Leisening's Theorem

Starting with the diagram from Pappus' Theorem (6.20), let O be the intersection of AB with A_1B_1 and let L1, L2, L3 be the intersections of OP and CC_1, OQ and BB_1, OS and AA_1 respectively. L1, L2, L3 are collinear.



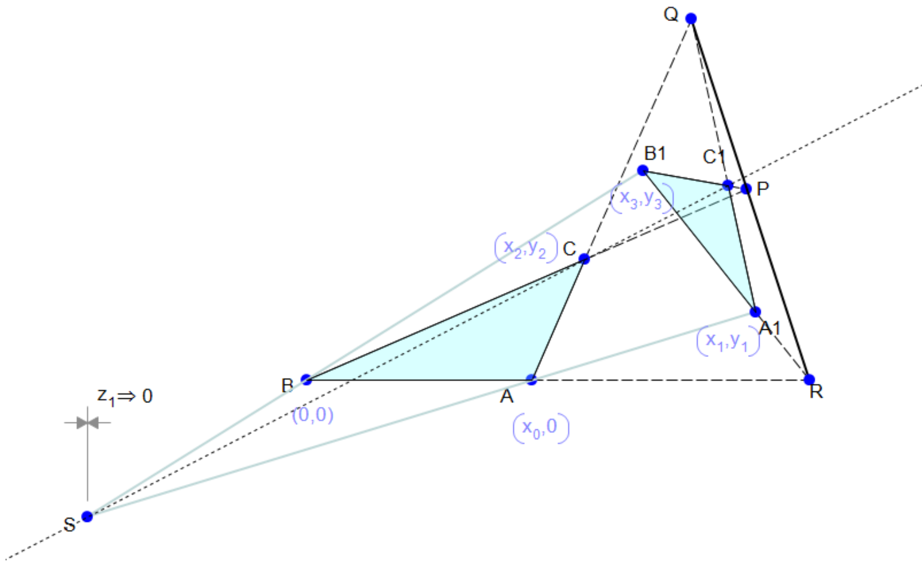
6.24 Desargues' Theorem

Given two triangles ABC, $A_1B_1C_1$, if the three lines AA_1, BB_1, CC_1 meet in a point S, define P,Q, and R to be the intersections of BC and B_1C_1, CA and C_1A_1, AB and A_1B_1 respectively. Then P, Q, R are collinear.



6.25 Converse of Desargues' Theorem

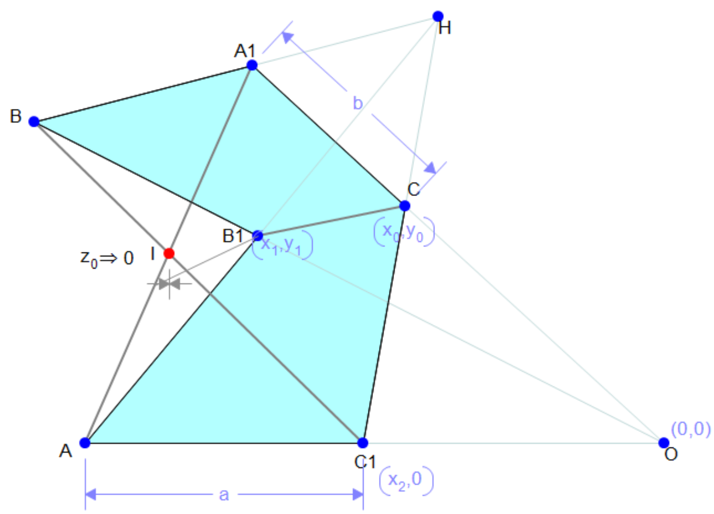
We constrain P,Q and R to be collinear and put point S at the intersection of AA_1 and BB_1. We measure its distance to the line CC_1.



2.5 Miscellaneous

6.26

In a hexagon $AC_1BA_1CB_1$, BB_1, C_1A, A_1C are concurrent and CC_1, A_1B, B_1A are concurrent. Show AA_1, B_1C, C_1B are also concurrent.



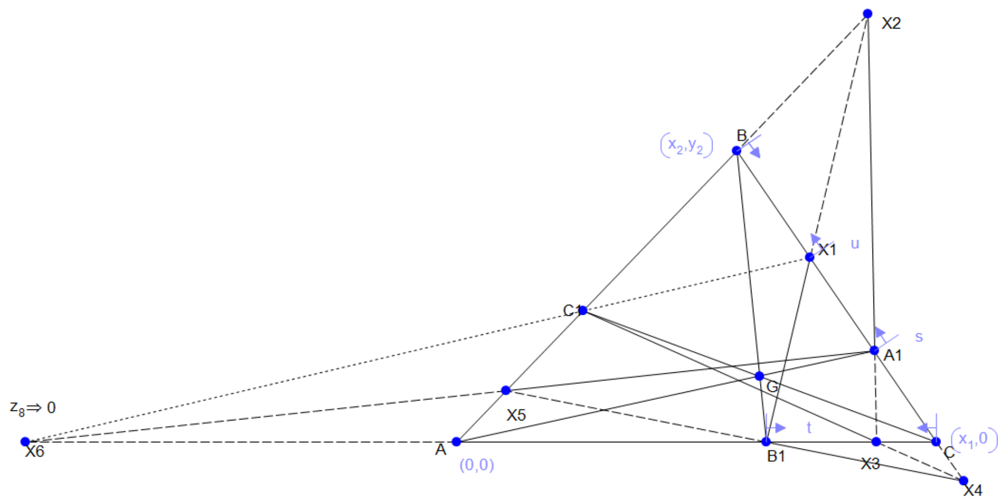
6.27 Nehring's Theorem

Let AA_1, BB_1, CC_1 be three concurrent Cevian lines for triangle ABC. Let X_1 be a point on BC,

$$X_2 = X_1B_1 \cap BA, X_3 = X_2A_1 \cap AC, X_4 = X_3C_1 \cap CB, X_5 = X_4B_1 \cap BA, X_6 = X_5A_1 \cap AC,$$

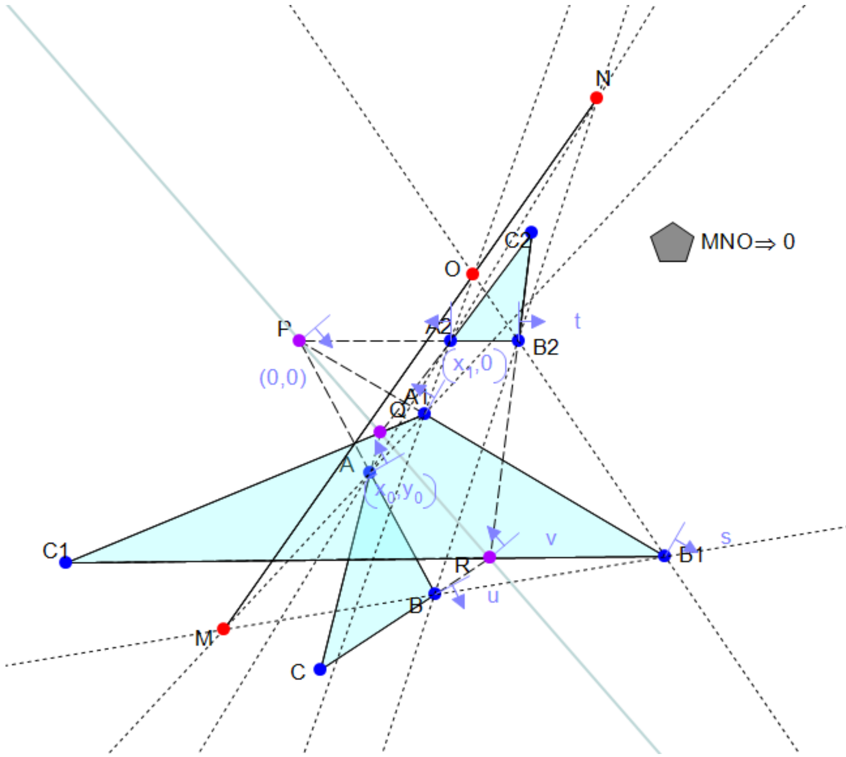
$X_7 = X_6C_1 \cap CB$

Show $X_7 = X_1$.



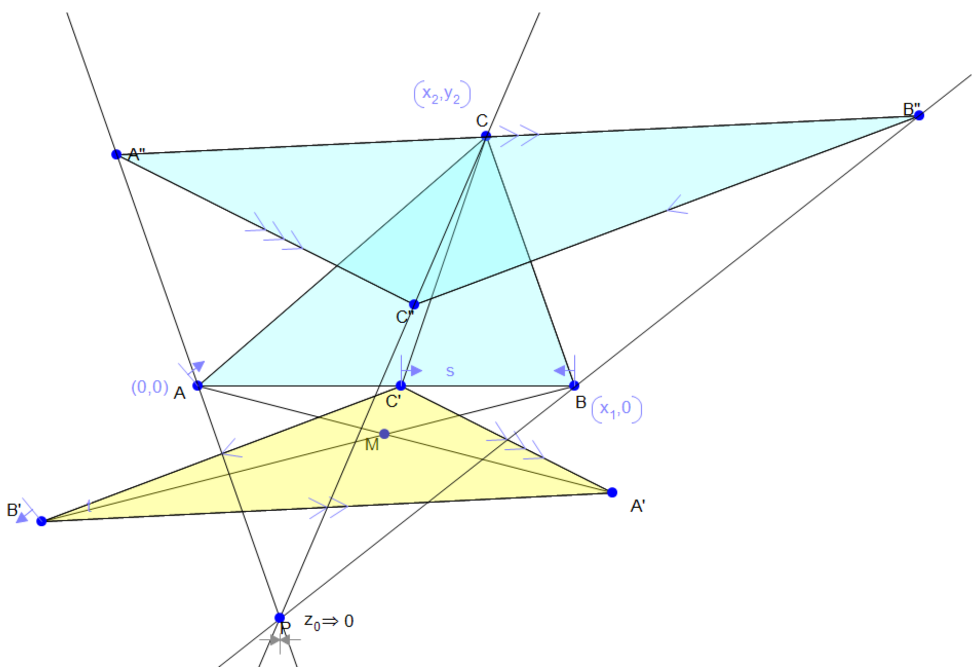
6.28

Let three triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$ be given such that lines AB, A_1B_1 intersect in point P, lines AC, A_1C_1, A_2C_2 intersect in point Q, lines BC, B_1C_1, B_2C_2 intersect in point R and P, Q and R are collinear. In view of Desargues' Theorem, the lines in each of the triads AA_1, BB_1, CC_1; AA_2, BB_2, CC_2; A_1A_2, B_1B_2, C_1C_2 intersect in a point. Prove that these three points are collinear.



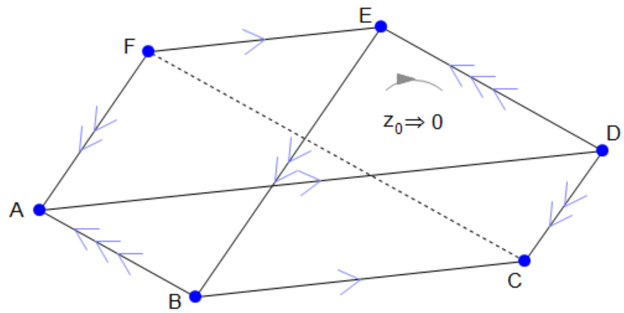
6.29

If (Q) is the cevian triangle of a point M for the triangle (P), show that the triangle formed by the parallels through the vertices of (P) to the corresponding sides of (Q) is perspective to (P)



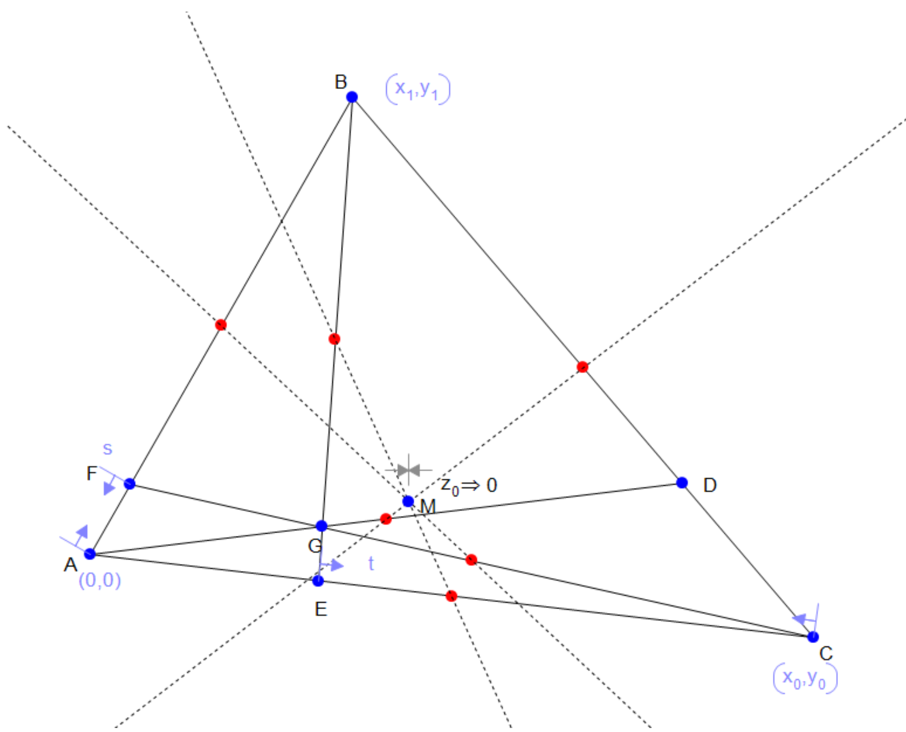
6.30

If a hexagon ABCDEF has two opposite sides BC and EF parallel to the diagonal AD and two opposite sides CF and FA parallel to the diagonal BE, while the remaining sides DE and AB are also parallel, then the third diagonal CF is parallel to AB



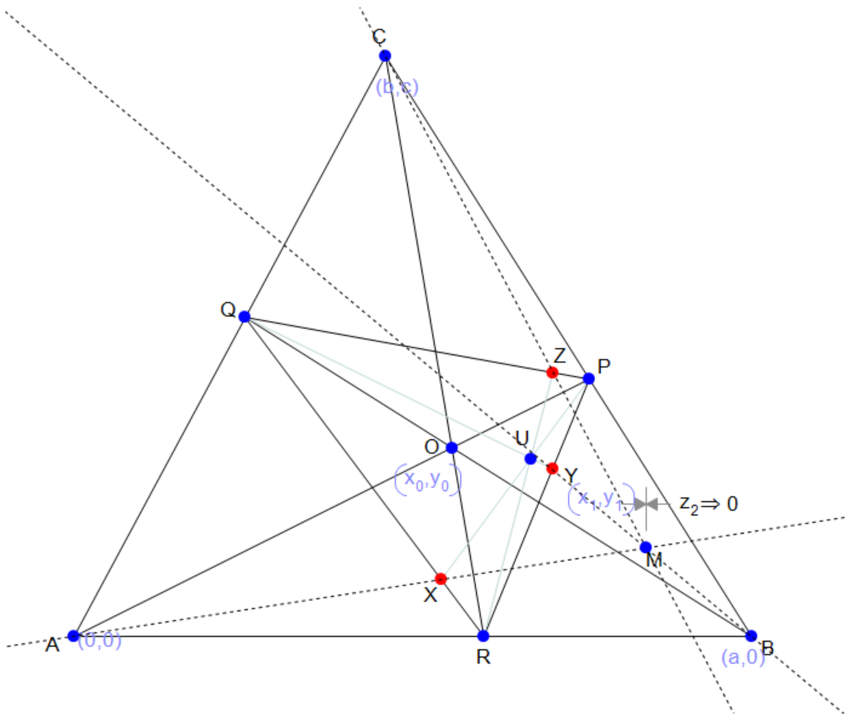
6.31

Prove that the lines joining the midpoints of three concurrent cevians to the midpoint of the corresponding sides of the given triangle are concurrent



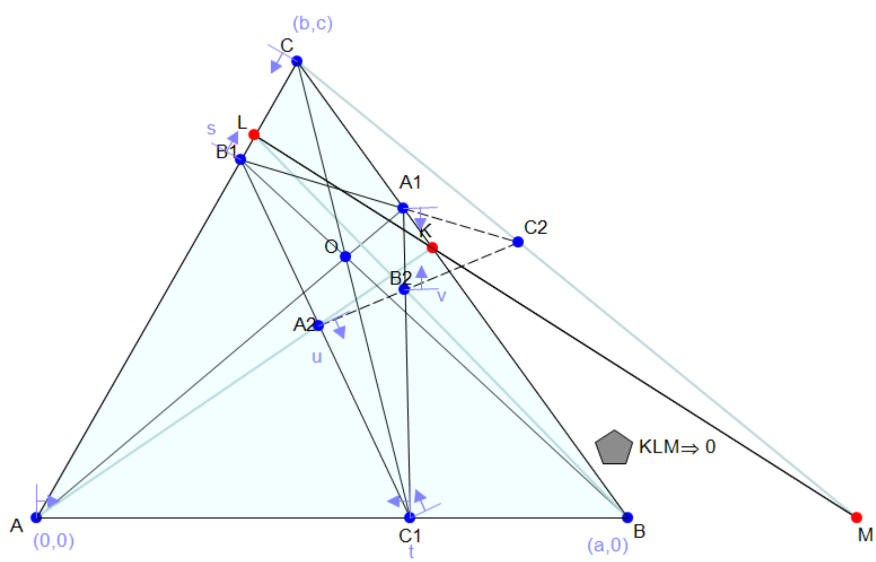
6.32

Let O and U be two points in the plane of the triangle ABC. Let AO, BO, CO intersect the opposite sides in P, Q, R. Let PU, QU, RU intersect QR, RP, PQ respectively in X, Y, Z. Show that AX, BY, CZ are concurrent.



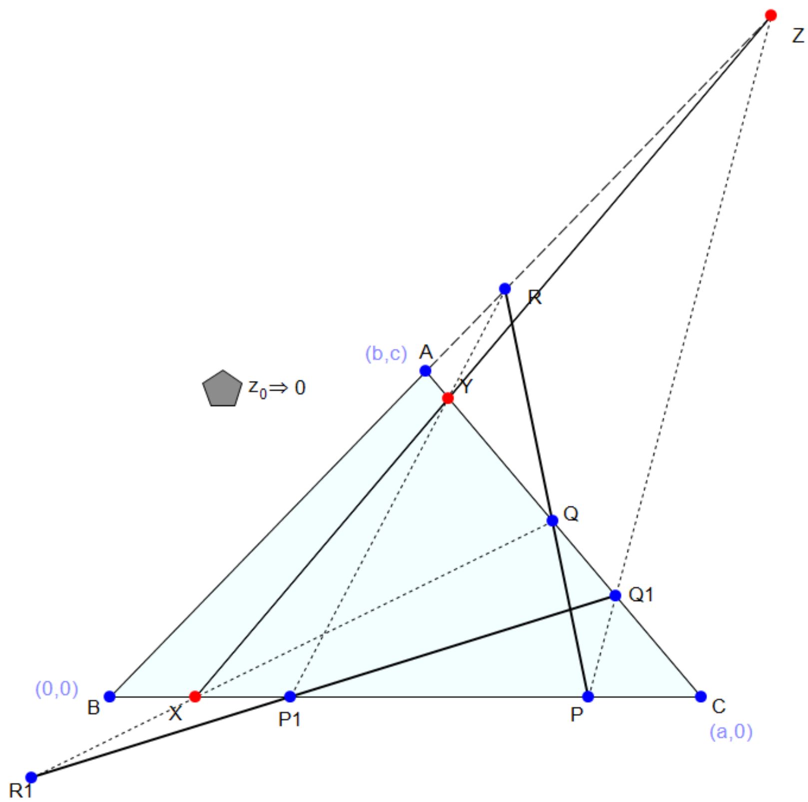
6.33

Let A_1, B_1, C_1 be the feet of Cevians through O in triangle ABC. Let A_2, B_2, C_2 be collinear points on the line BC, AC, AB. The points of intersection of the lines AA_2, BB_2, CC_2 with the opposite sides of ABC are collinear.



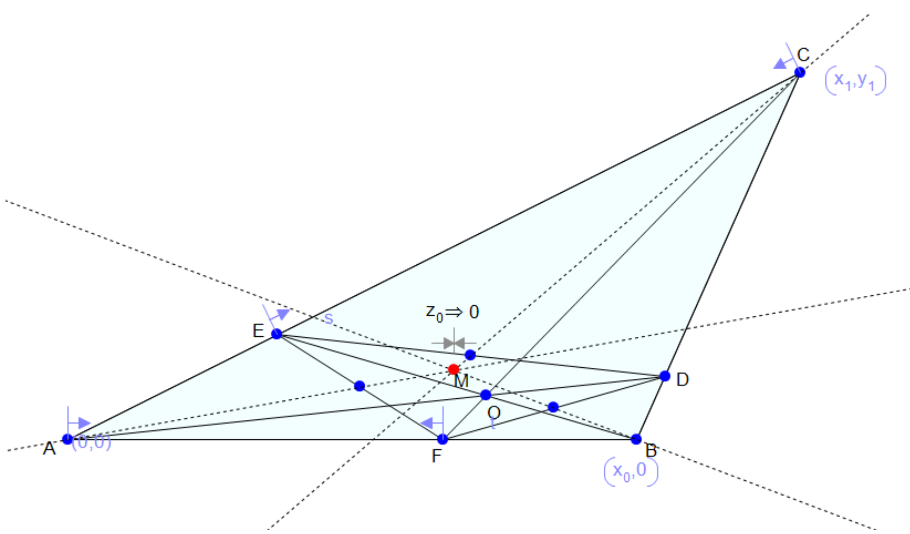
6.34

The sides BC, CA, AB of a triangle ABC are met by two transversals PQR, $P_1Q_1R_1$. Let X be the intersection of BC and QR_1, let Y be the intersection of CA and RP_1, let Z be the intersection of AB and PQ_1. Show X,Y, Z are collinear.



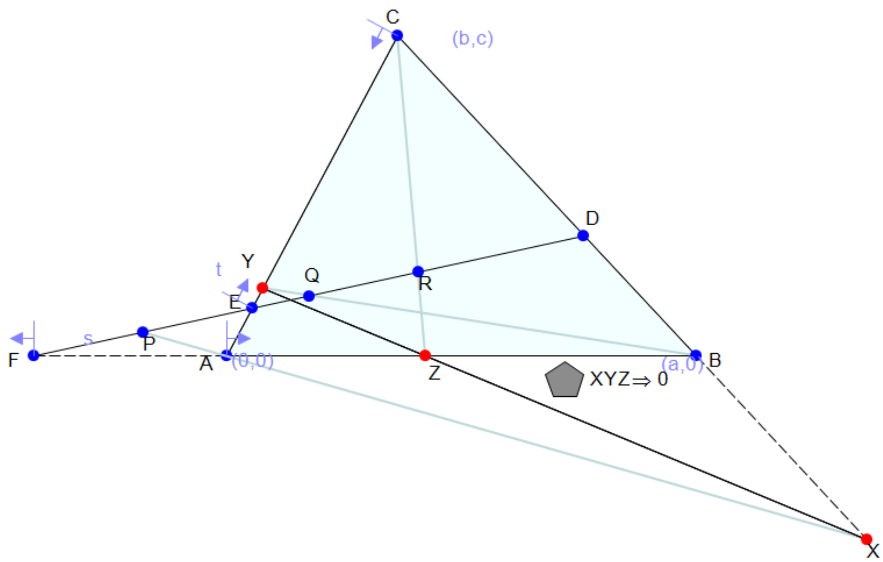
6.35

Through the vertices of a triangle ABC lines are drawn intersecting in O and meeting the opposite sides in D, E, F. Prove that the lines joining A,B,C to the midpoints of EF, FD, DE are concurrent.



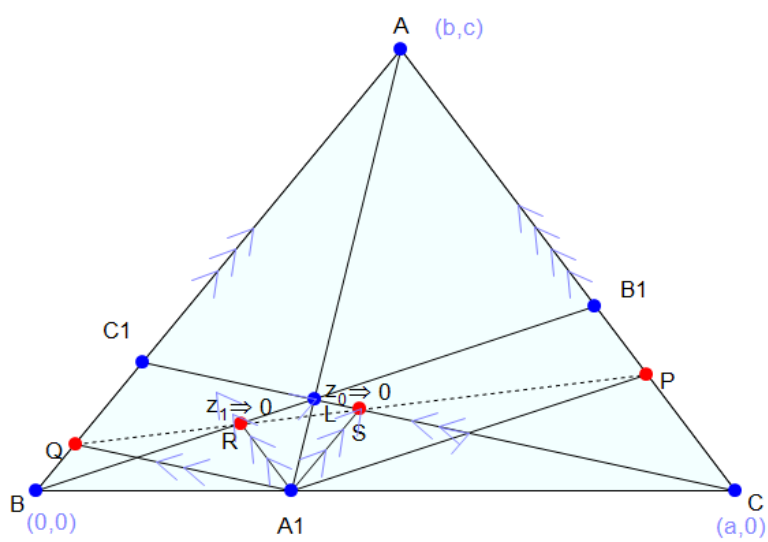
6.36

A transversal cuts the sides BC, CA, AB of triangle ABC in D, E, F. P, Q, R are the midpoints of EF, FD, DE, and AP, BQ, CR intersect BC, CA, AB in X, Y, Z. Show that X, Y, Z are collinear.



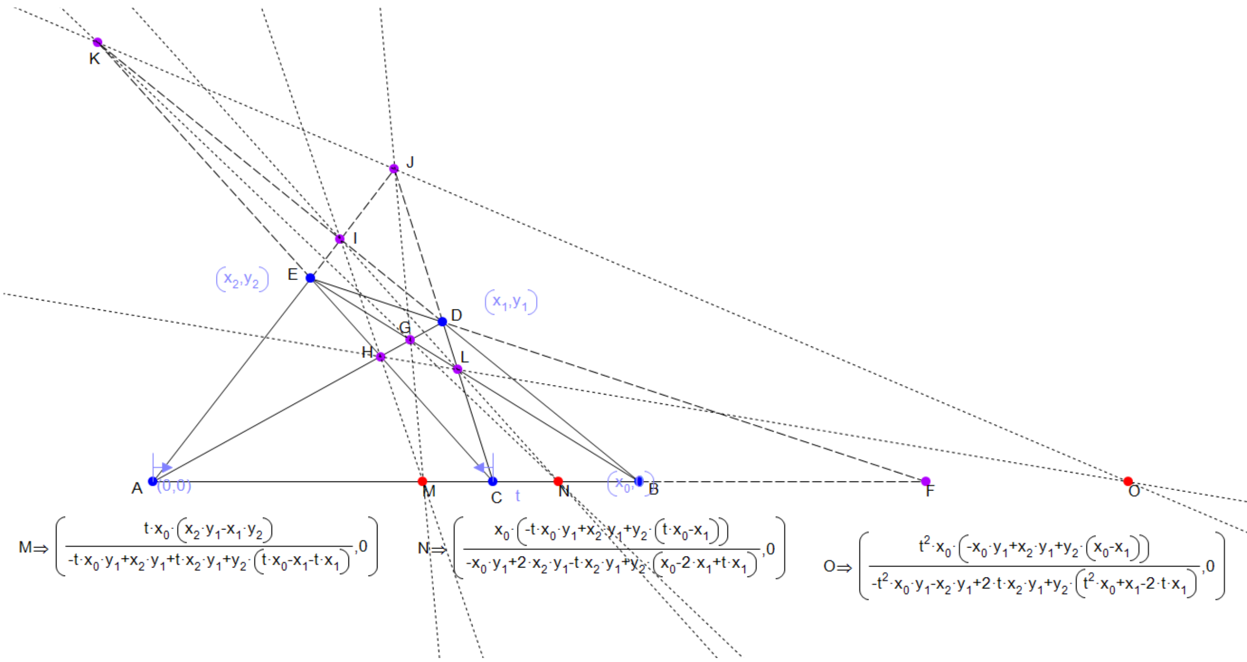
6.37

The lines AL, BL, CL joining the vertices of a triangle ABC to a point L meet the opposite sides in A_1, B_1, C_1. The parallels through A_1 to BB_1 CC_1 meet AC, AB in P, Q and the parallels through A_1 to AC, AB meet BB_1, CC_1 in R, S. Show that P, Q, R, S are collinear.



6.38

Given 5 points A, B, C, D, E with A, B, C collinear. New lines and points of intersection are formed as in the figure. We show (1) AB, GJ, HI are collinear (2) AB, GK and IL are collinear, (3) AB, HL and JK are collinear (do we mean concurrent?).



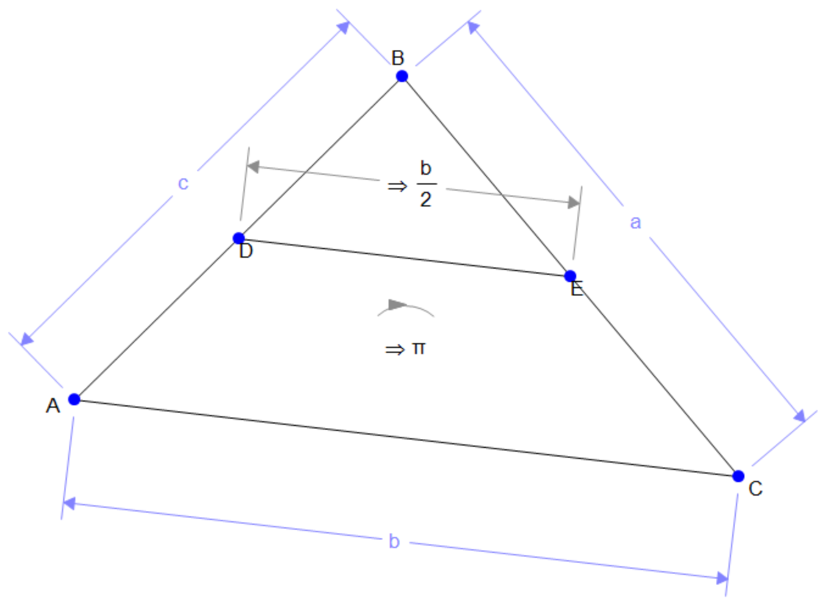
We have forced AB to lie on the x axis. M, N and O are defined as the intersection of the respective doted lines. Observing that their y coordinates are zero shows that the pairs of dotted line are concurrent with AB.

3 Triangles

3.1 Medians and Centroids

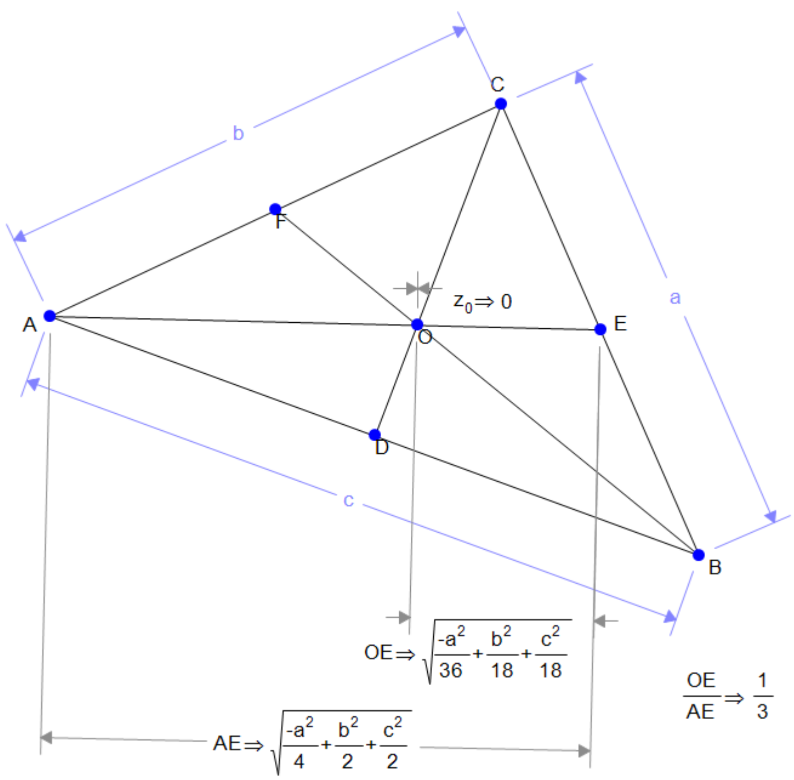
6.39

The line joining the midpoints of two sides of a triangle is parallel to the third side and is equal to one half the length.



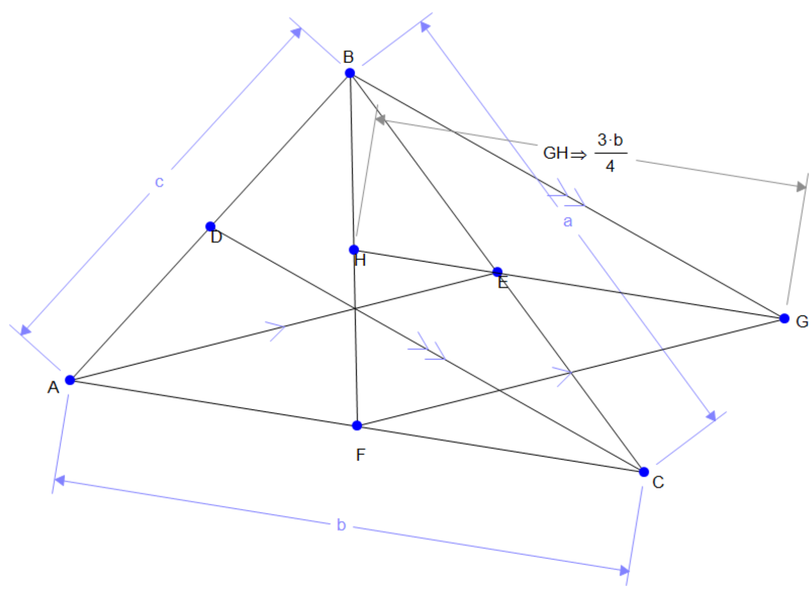
6.40 Theorem of Centroid

The three medians of a triangle meet in a point and each median is trisected by this point.



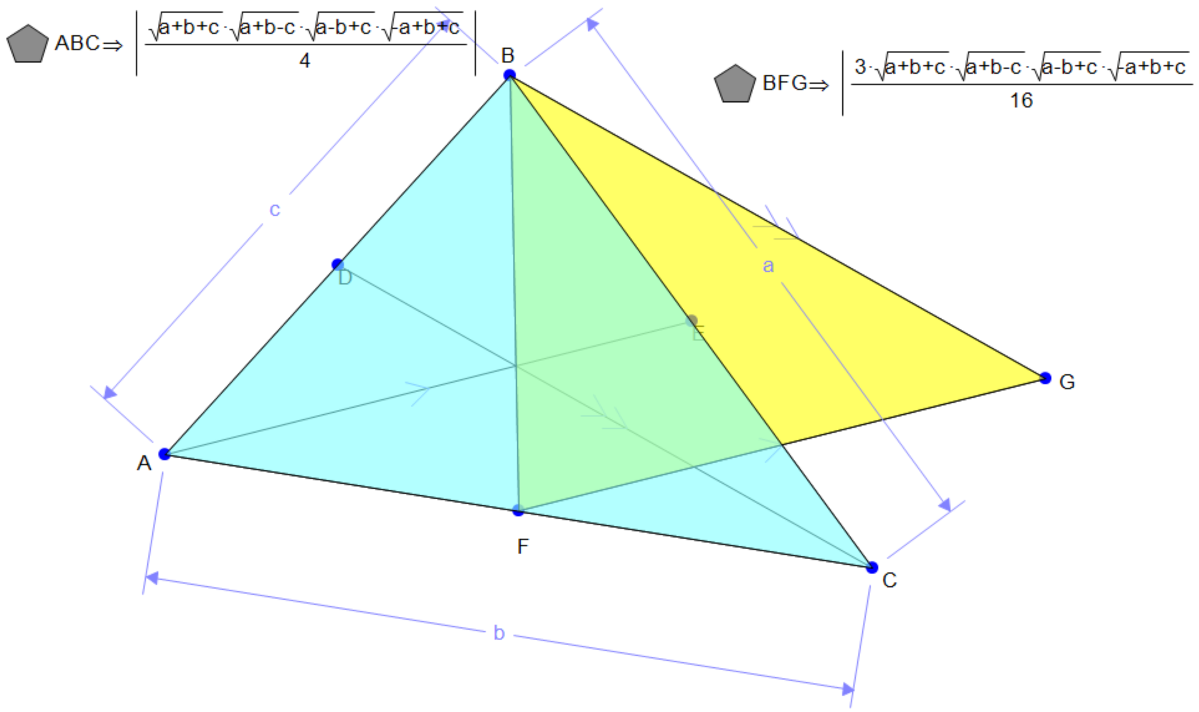
6.41

With the medians of a triangle a new triangle is constructed. The medians of the second triangle are equal to three quarters of the respective sides of the given triangle.



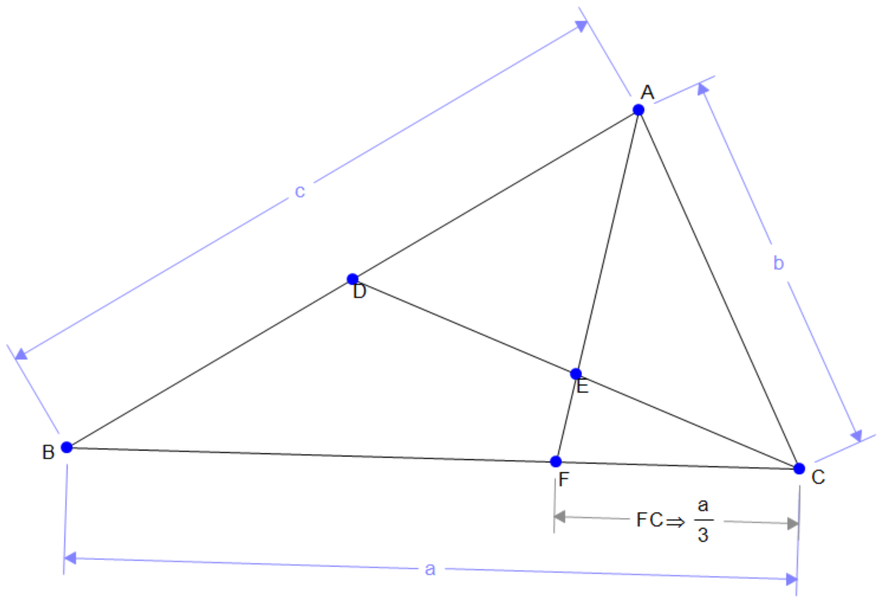
6.42

The area of the triangle having for sides the medians of a triangle is equal to three quarters the area of the given triangle.



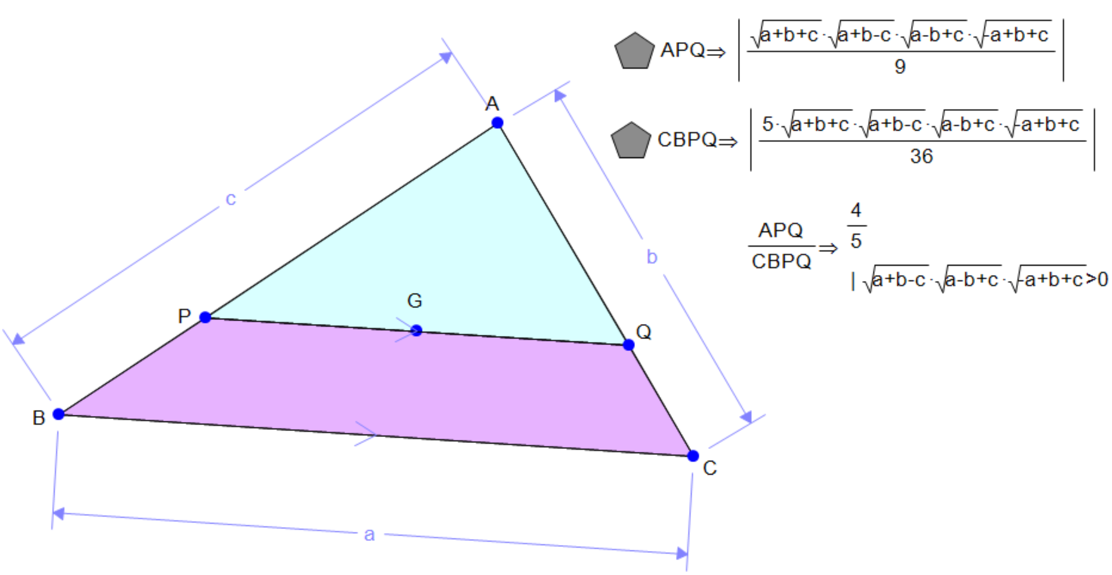
6.43

Show that the line joining the midpoint of a median to a vertex of the triangle trisects the side opposite the vertex considered.



6.44

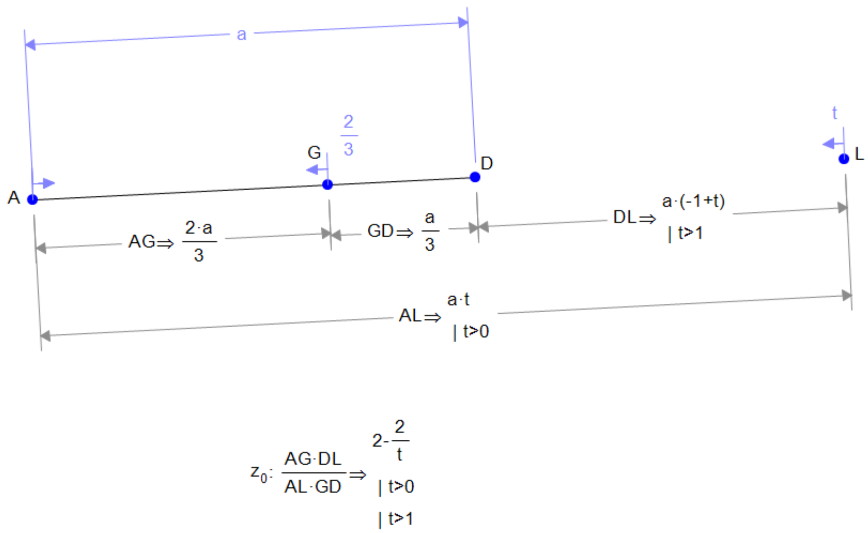
Show that a parallel to a side of a triangle through the centroid divides the area of the triangle into two parts, in the ratio of 4:5



6.45

If L is the harmonic conjugate of the centroid G of a triangle ABC with respect to the ends A, D of the median AD, show that LD = AD

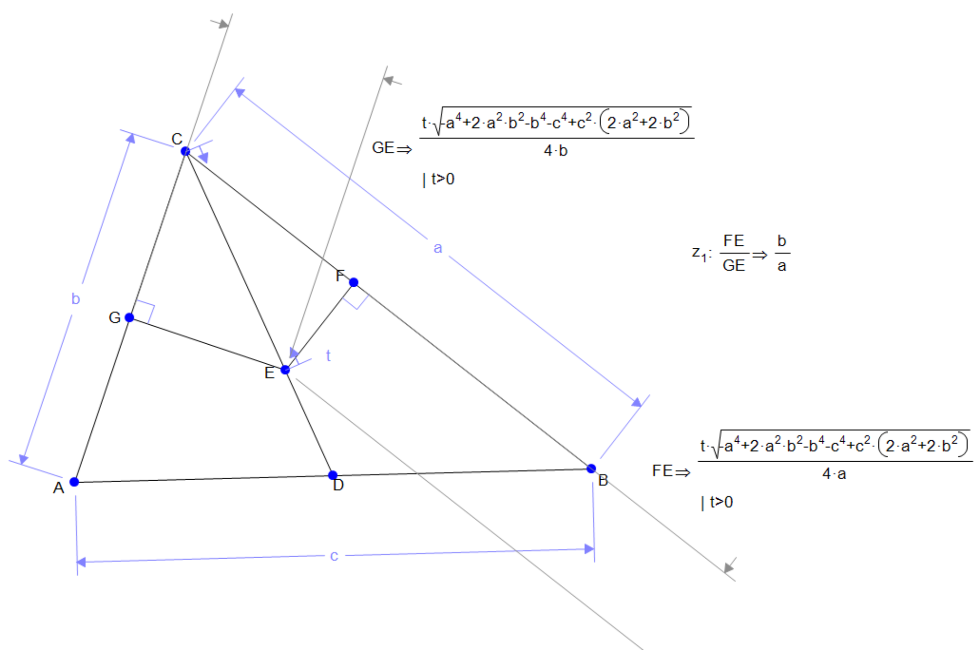
First we need to work out how to create the harmonic conjugate of the centroid, as this is not a Geometry Expressions function. The centroid is 2/3 of the way down the median. We need to see what location on the median the harmonic conjugate sits at.



Parametric location t=2 will do the trick… and will also make AD = BL.

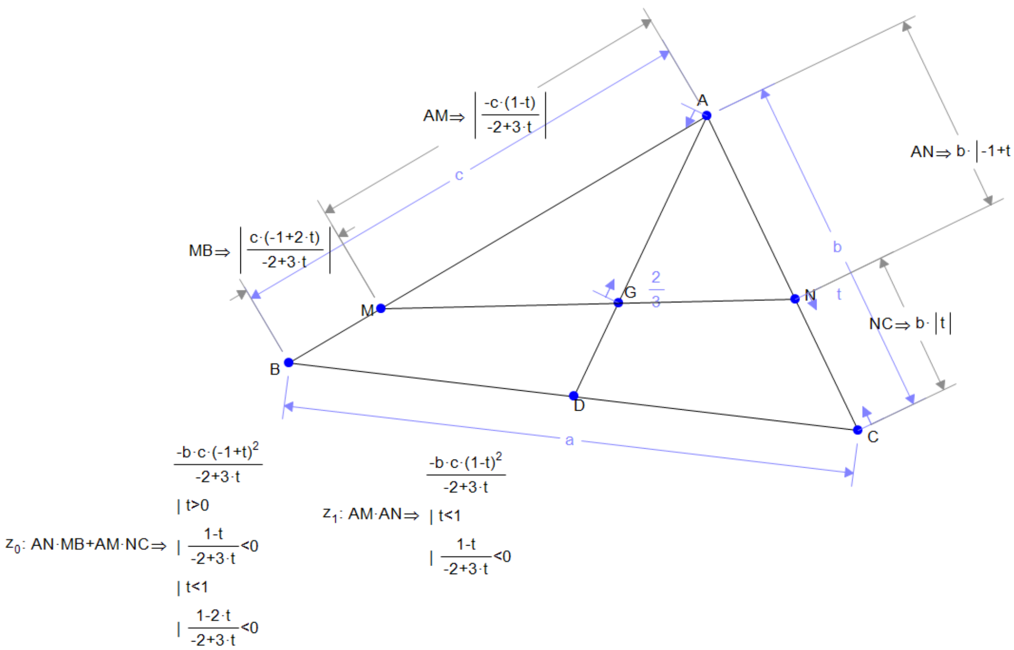
6.46

Show that the distances of a point on a median of a triangle from the sides including the median are inversely proportional to these sides.



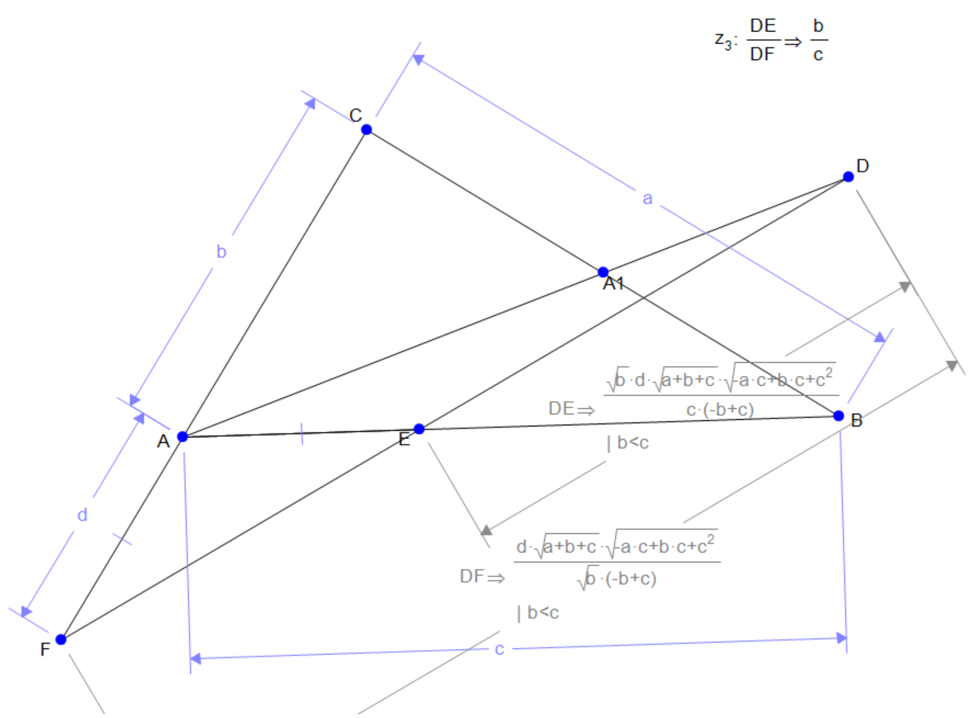
6.47

Show that, if a line through the centroid G of the triangle ABC meets AB in M and AC in N we have AN*MB+AM*NC=AM*AN



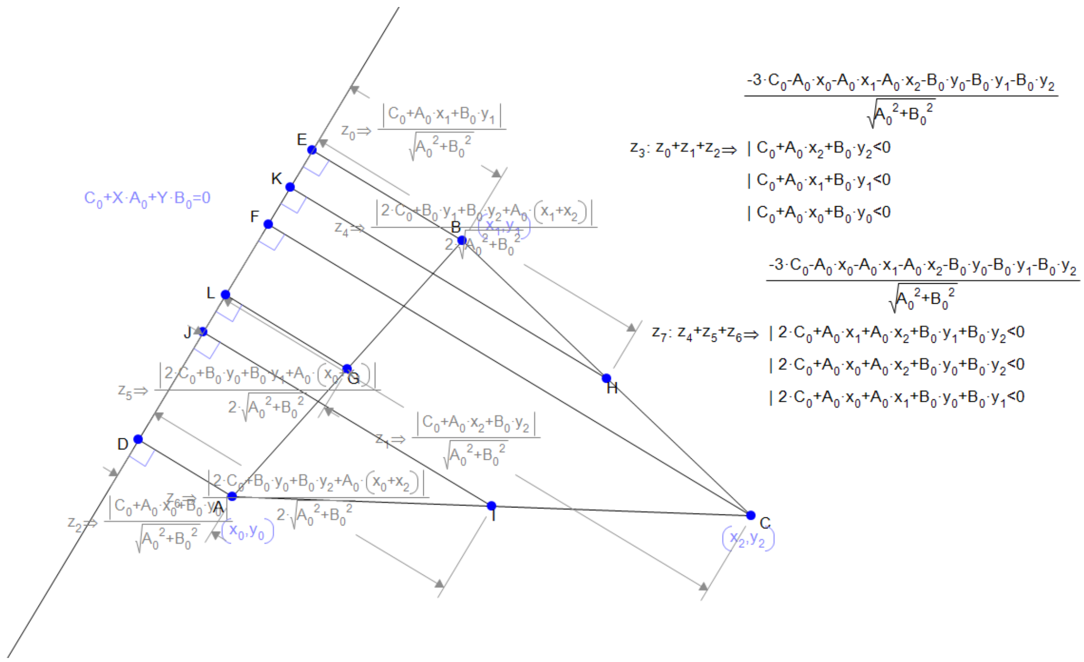
6.48

Two equal segments AE, AF are taken on the sides AB, AC of the triangle ABC. Show that the median issued from A divides EF in the ratio of the sides AC, AB



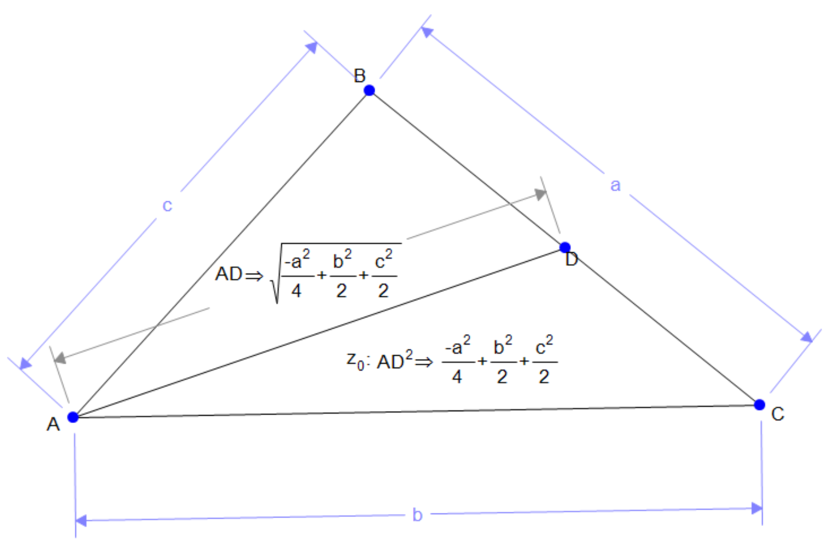
6.49

Show that the algebraic sum of the distances of the vertices of a triangle to a line is the same as the sum of the distances of the midpoints to that line



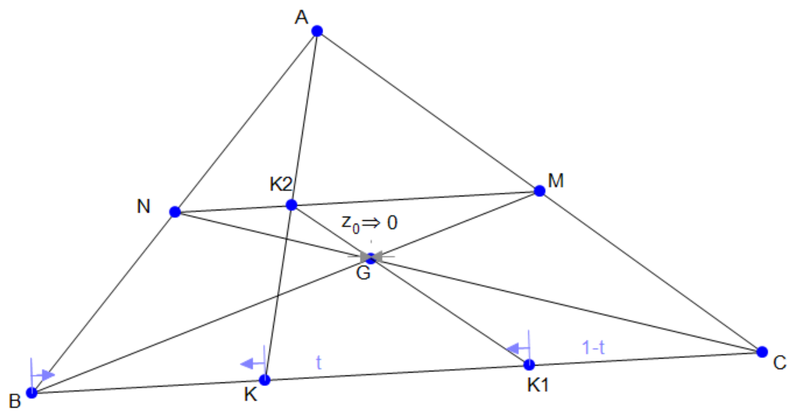
6.50

Compute the square of the lengths of the medians



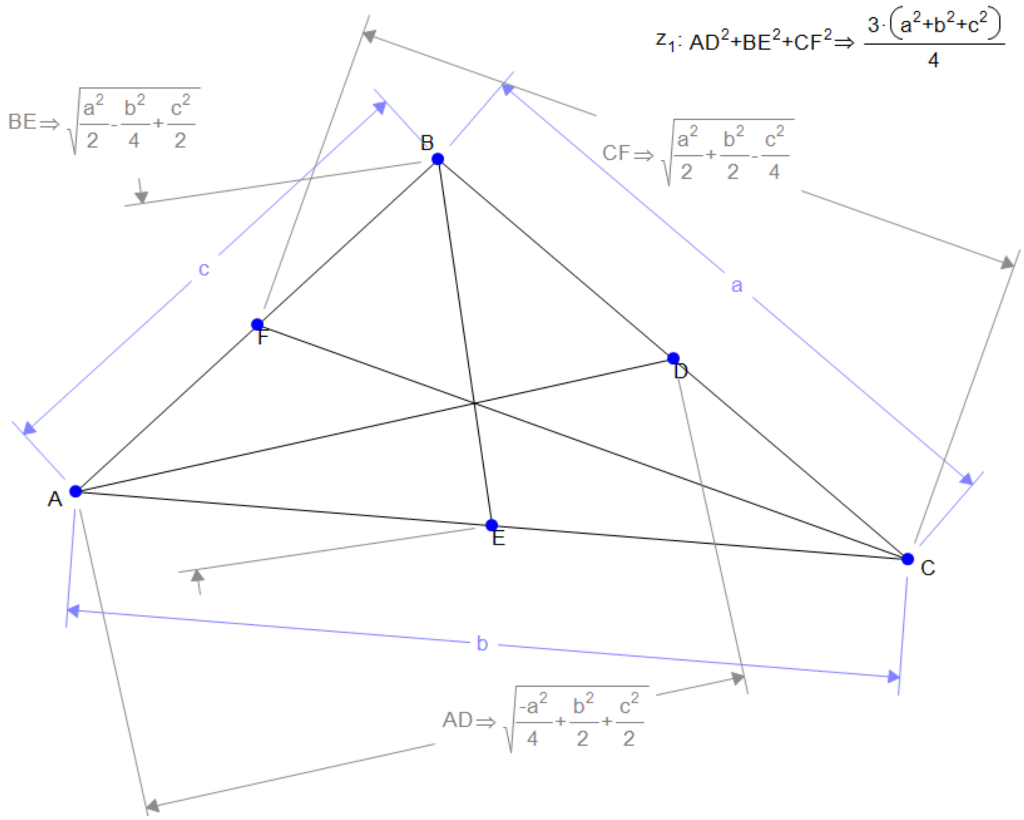
6.51

If K, K1 are two isotomic points on the side BC of triangle ABC, and the line AK meets the line MN in K2, where N and M are the midpoints of AB and AC. Show that K1K2 passes through the centroid G of ABC



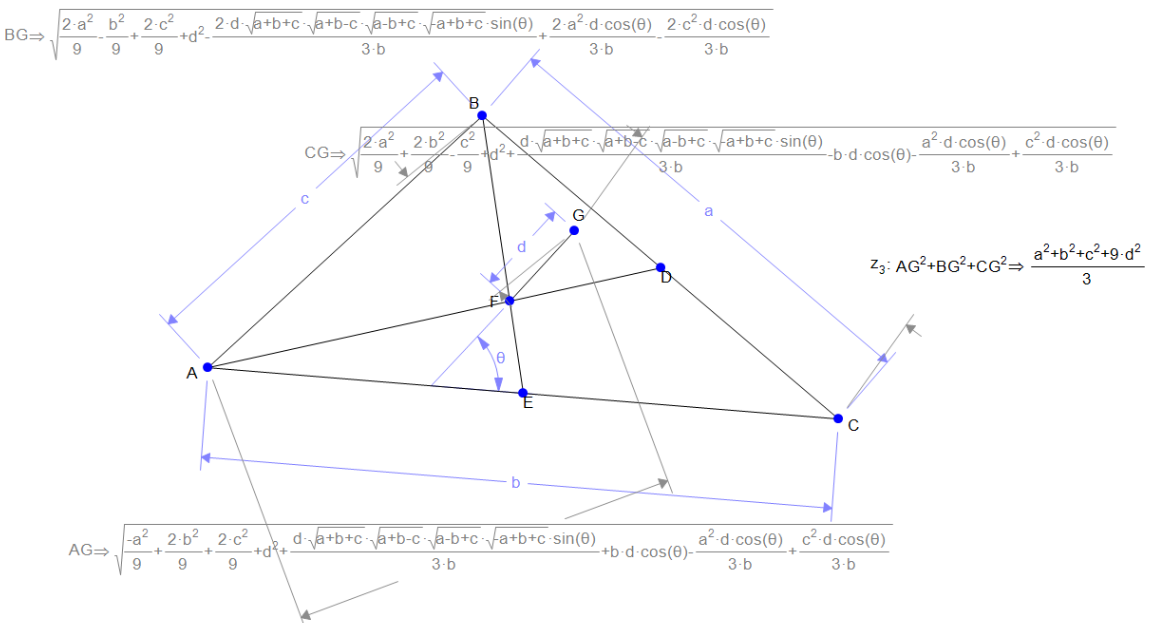
6.52

The sum of the squares of the medians is equal to ¾ the sum of squares of the sides of the original triangle



6.53

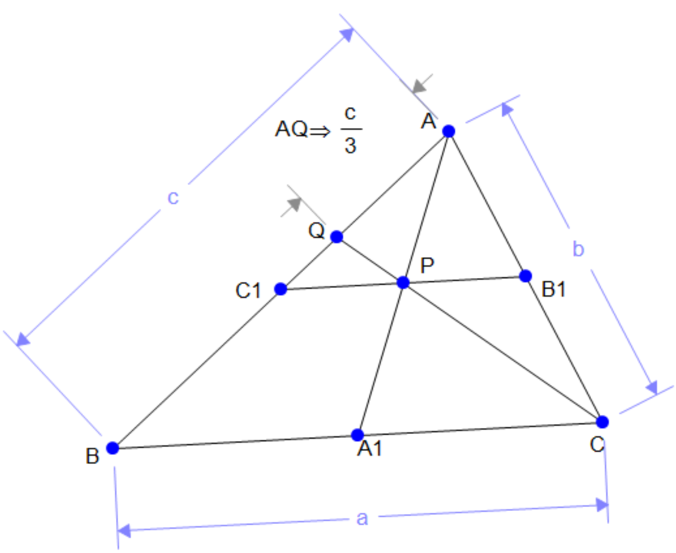
If two points are equidistant from the centroid of a triangle, the sums of the squares of their distances from the vertices of the triangle are equal.



The sum of squares is independent of θ, hence the result.

6.54

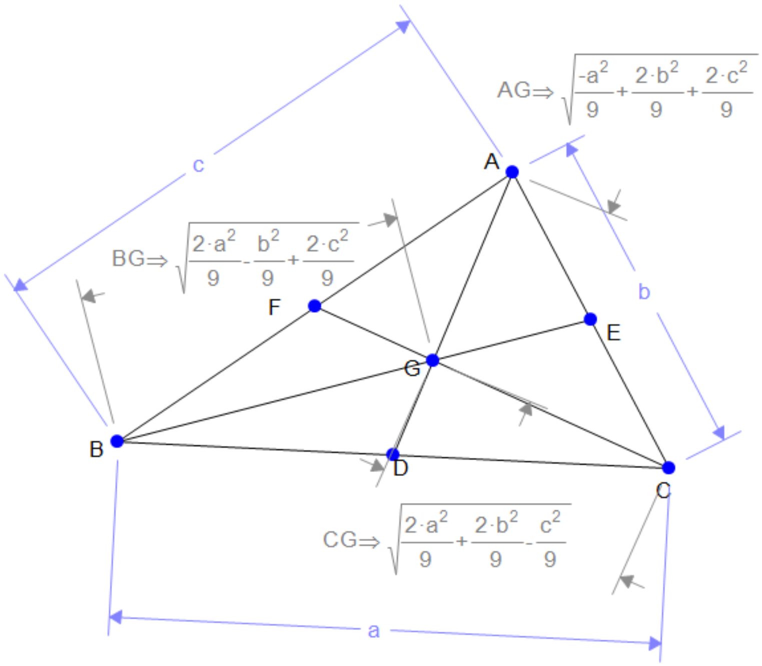
The median AA_1 of triangle ABC meets the side B_1C_1 in P, and CP meets AB in Q. Show that AB=3AQ.



6.55

The sum of squares of the distances of the centroid of a triangle from the vertices is equal to one third the sum of squares of the sides.

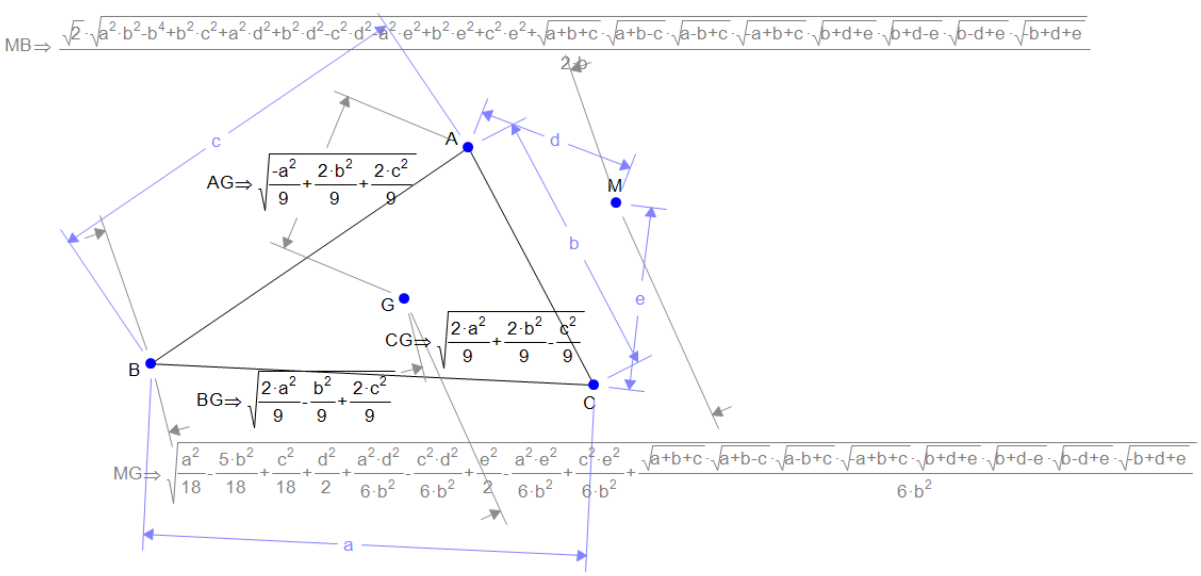
$$z_0: AG^2+BG^2+CG^2 \Rightarrow \frac{a^2+b^2+c^2}{3}$$



6.56

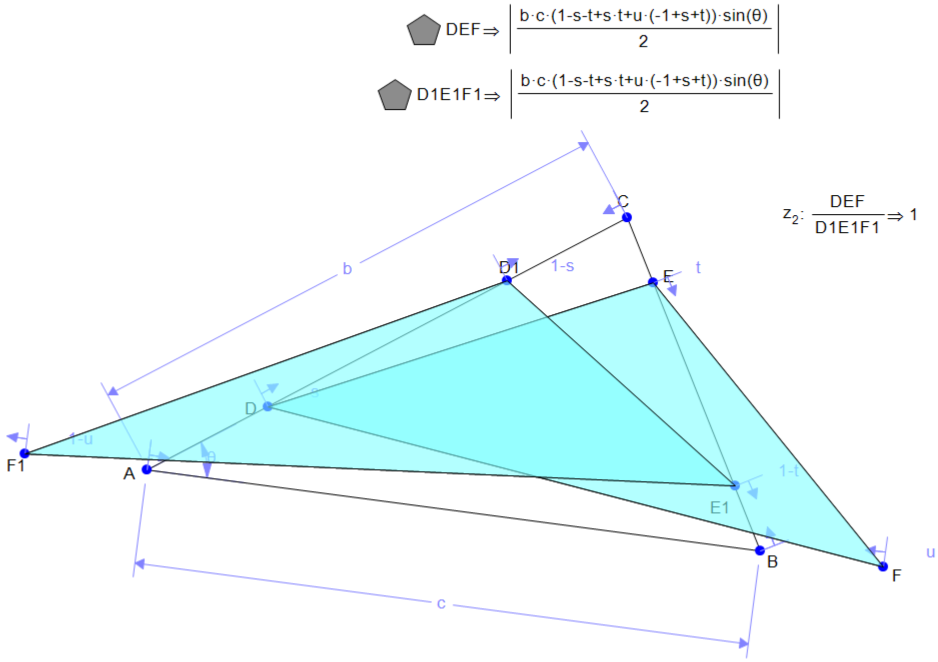
If M is any point in the plane of the triangle ABC and G is the centroid of ABC, we have $MA^2+MB^2+MC^2=GA^2+GB^2+GC^2+3MG^2$

$$z_0: MB^2-3\cdot MG^2+d^2+e^2 \Rightarrow \frac{a^2+b^2+c^2}{3} \qquad z_3: AG^2+BG^2+CG^2 \Rightarrow \frac{a^2+b^2+c^2}{3}$$



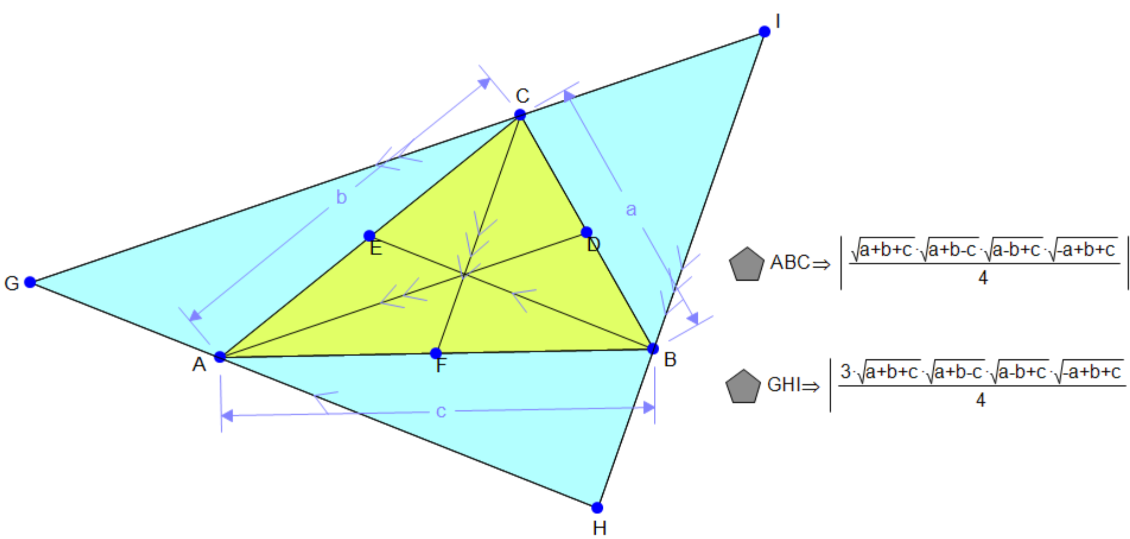
6.57

If the sides D,D_1; E,E_1; F,F_1 are isotomic on the sides BC, CA, AB of triangle ABC, the areas of the triangles DEF, $D_1E_1F_1$ are equal.

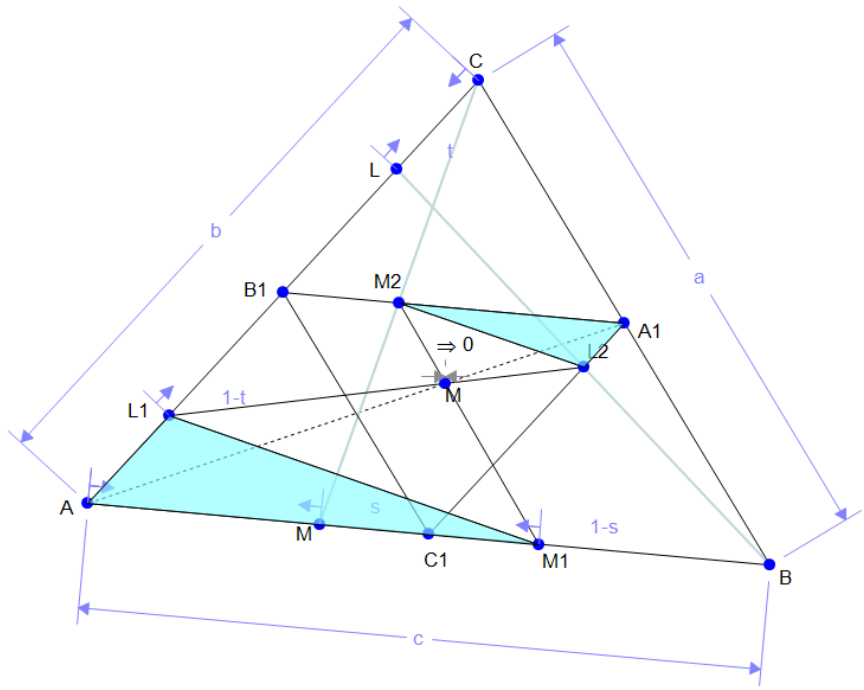


6.58

Show that the parallels through the vertices A, B, C of the triangle ABC to the medians of this triangle issued from the vertices B, C, A respectively form a triangle whose area is three times the area of the original triangle



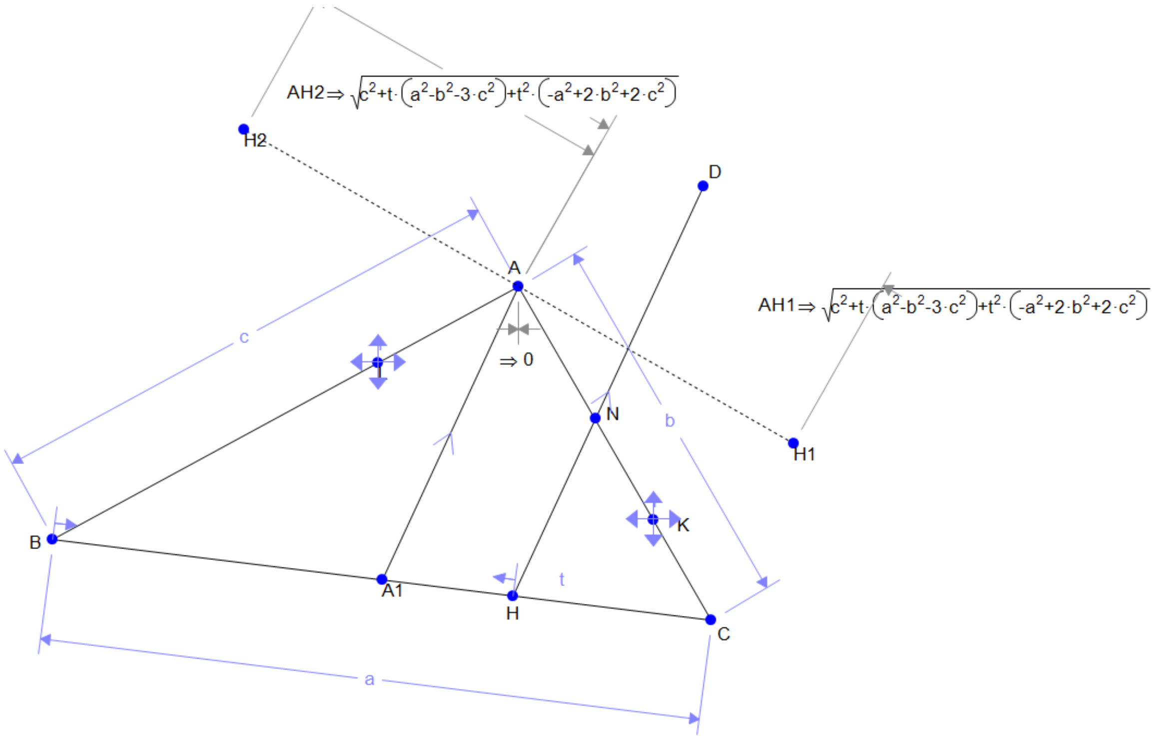
Let L, L_1 and M,M_1 be two pairs of isotomic points on the two sides AC, AB of triangle ABC, and L_2,M_2 the traces of the lines BL, CM on the sides A_1C_1, A_1B_1 of the medial triangle $A_1B_1C_1$ of ABC. Show that the triangles AL_1M_1, $A_1L_2M_2$ are homothetic.



M is the intersection of L_1L_2 and M_1M_2. We show that M lies on AA_1

6.60

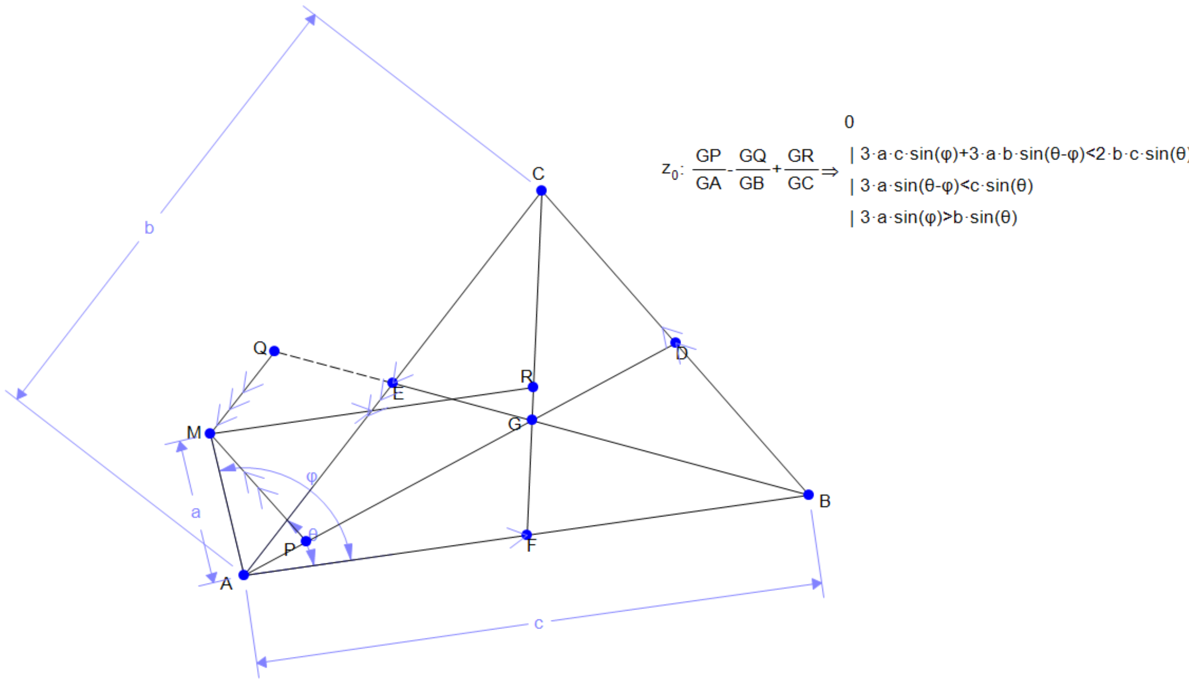
A parallel to the median AA_1 of the triangle ABC meets BC, CA, AB in the points H, N, D. Prove that the symmetries of H with respect to the midpoints of NC, BD are symmetrical with respect to the vertex A



We see that AH1 and AH2 are the same length. Also A lies on the line between H1 and H2.

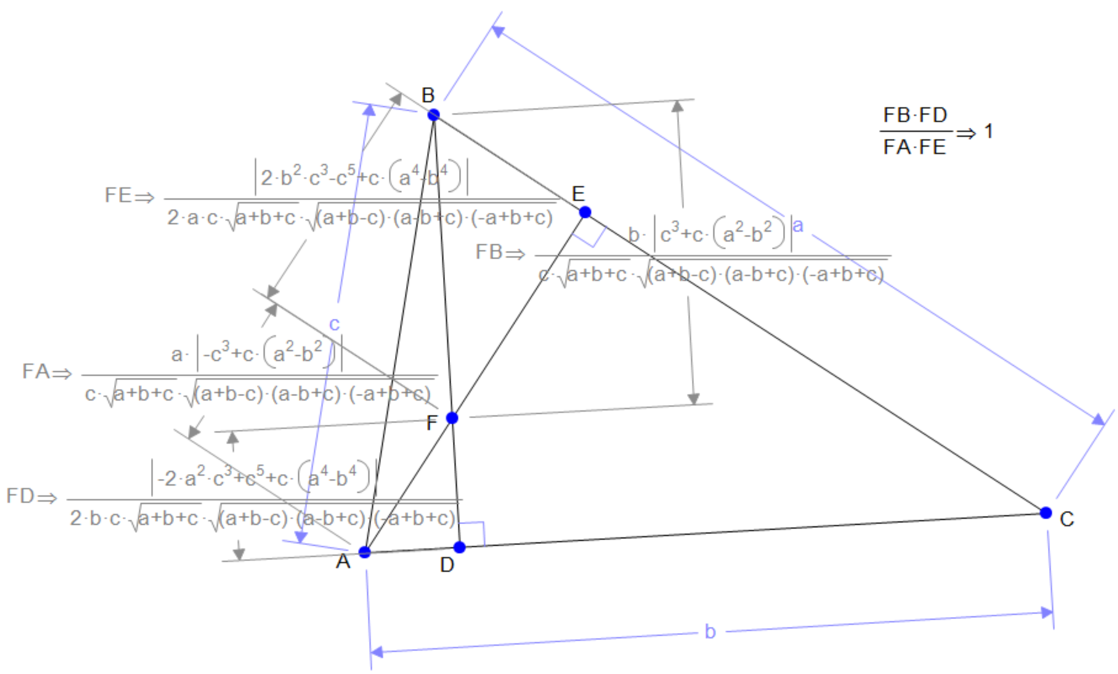
6.61

The parallels to the sides of a triangle ABC through the same point M meet the respective medians in the points P, Q, R. Prove that we have GP/GA = GQ/GB +GR/GC



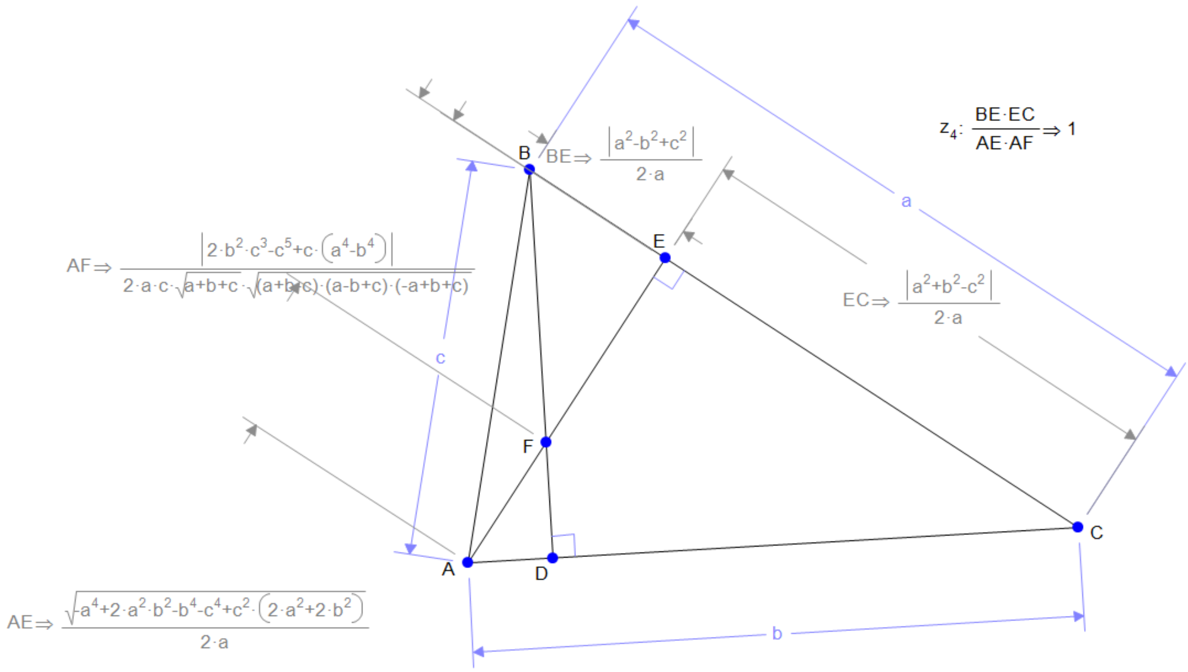
6.63

In a given triangle, the three products of the segments into which the orthocenter divides the altitudes are equal.



6.64

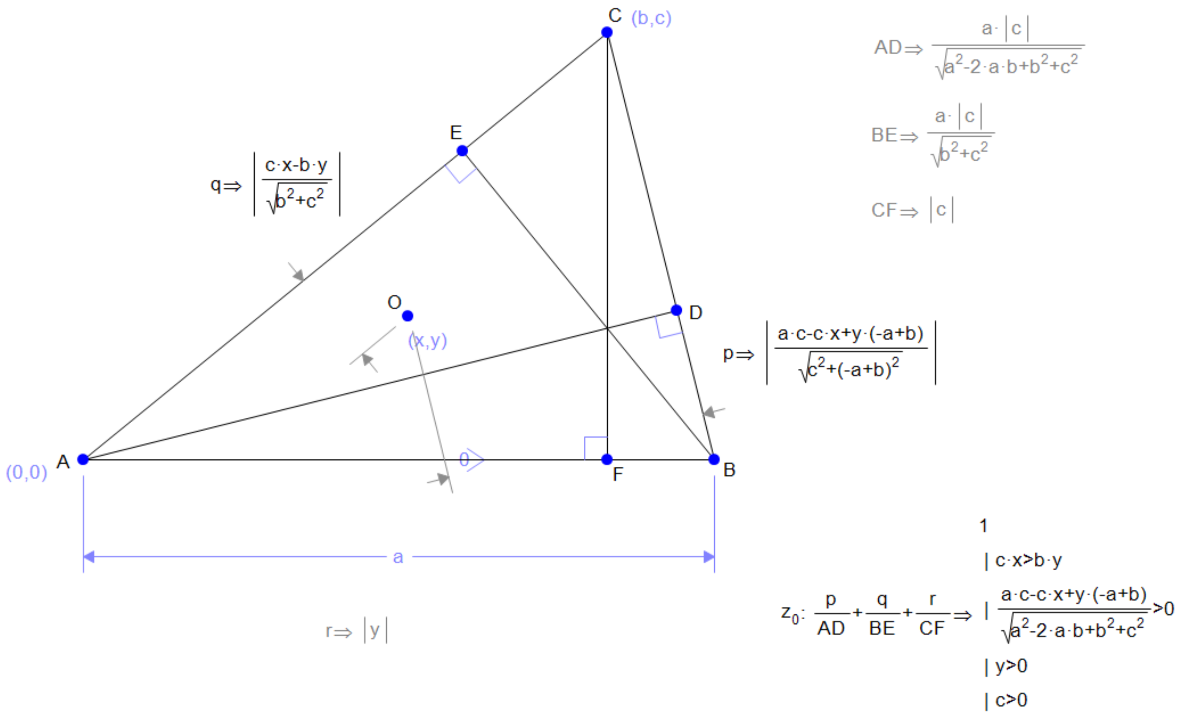
The product of the segments into which the side of a triangle is divided by the foot of the altitude is equal to this altitude multiplied by the distance of the side from the orthocenter.



6.65

If p,q, r are the distances of a point inside a triangle ABC from the sides of the triangle, show that

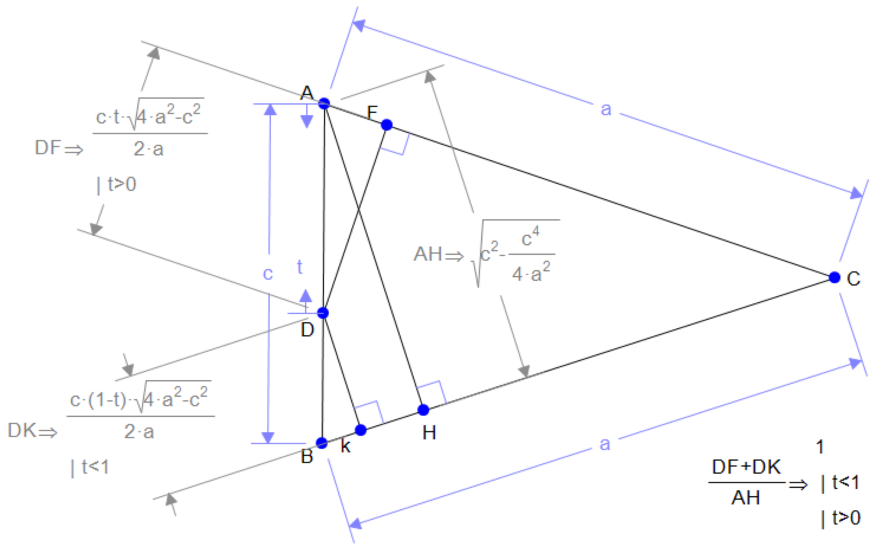
$\frac{p}{h_a}+\frac{q}{h_b}+\frac{r}{h_c}$=1.



Specifying the triangle by coordinates allows us to give the location of O by coordinate.

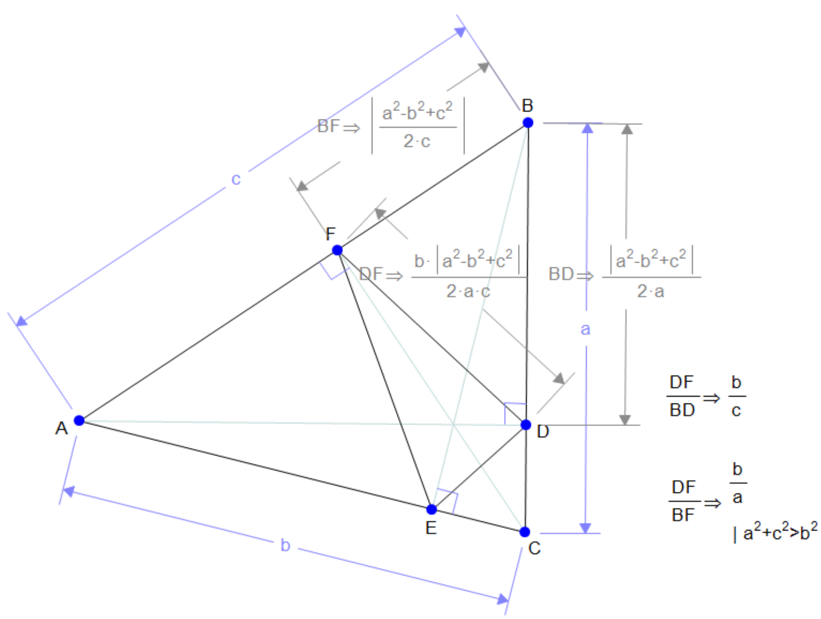
6.66

Show that the sum of the distances of a point on the base of an isosceles triangle to its two sides is equal to the altitude on that side.



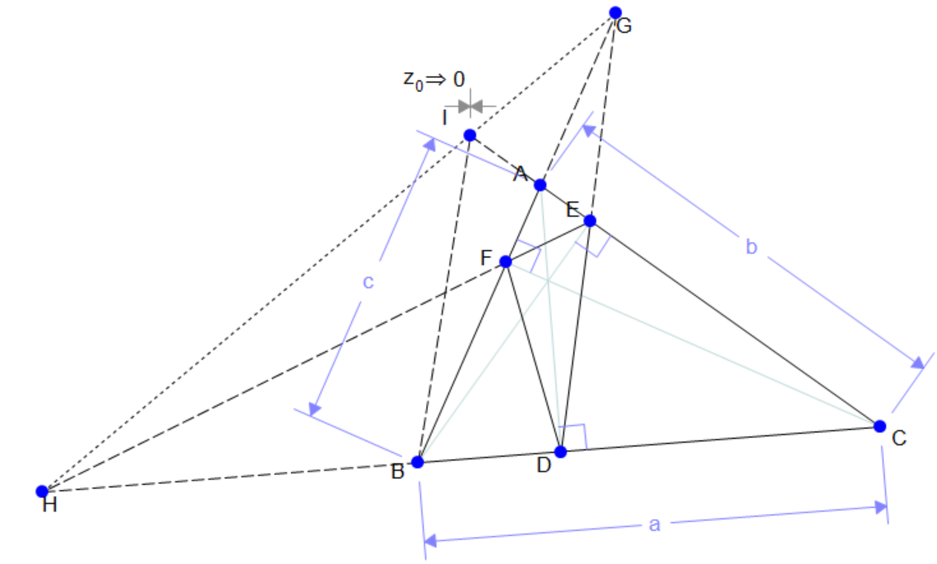
6.67

The three triangles cut off by the sides of its orthic triangle are similar to the given triangle.



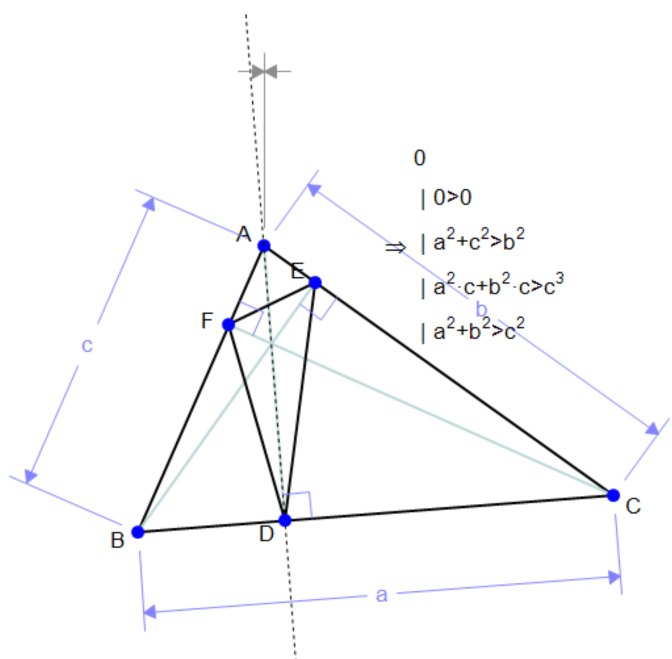
6.68

The sides of the orthic triangle meet the sides of the given triangle in three collinear points.



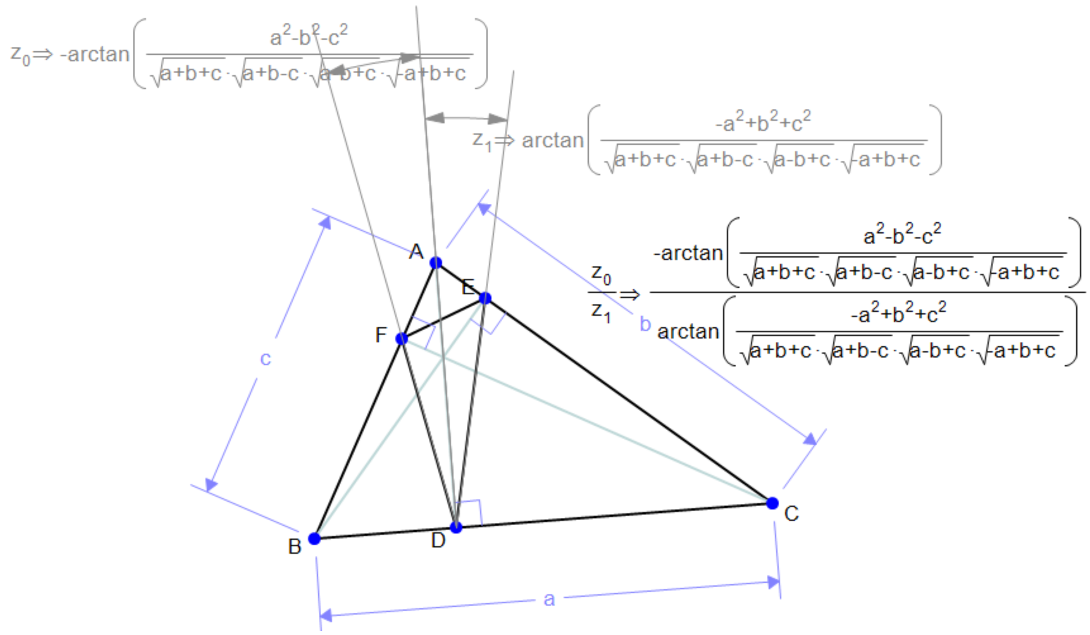
6.69

The altitudes of a triangle bisect the internal angles of its orthic triangle.



We can show that point A lies on the bisector of angle FDE, hence the altitude is the bisector.

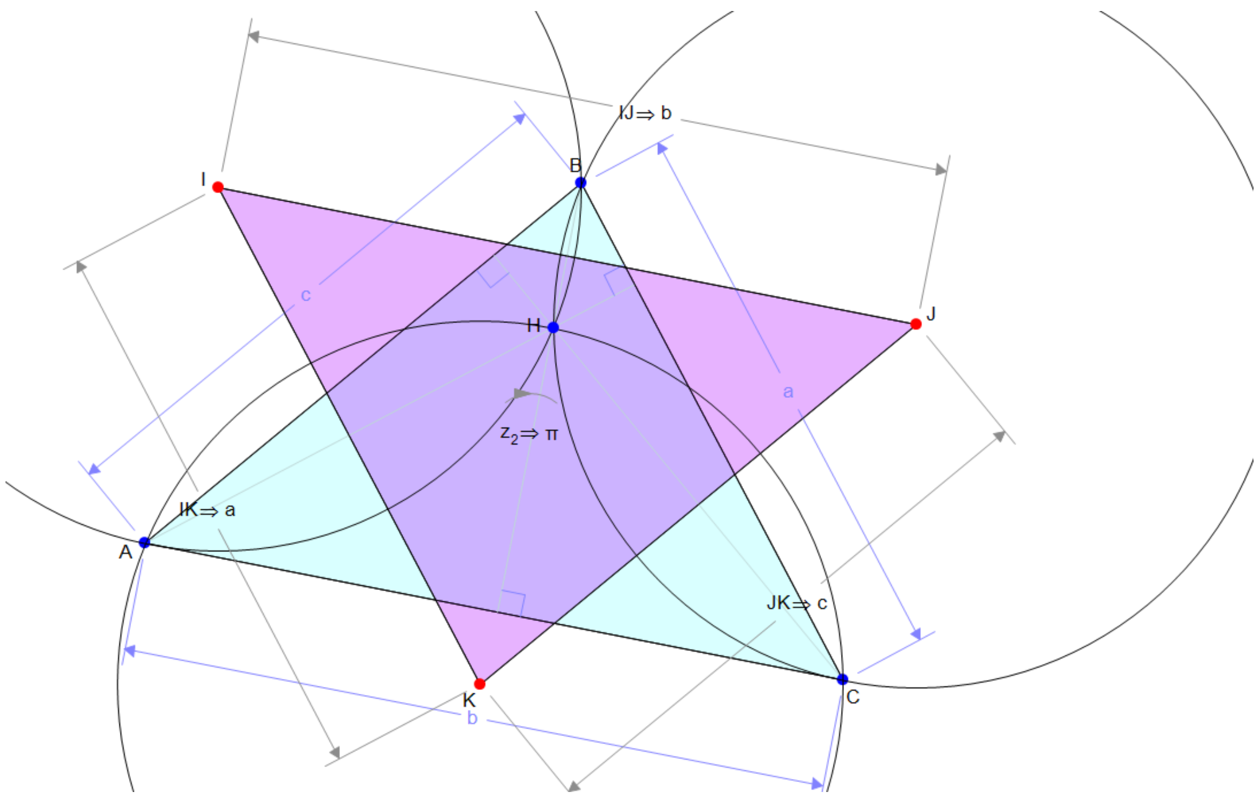
A simpler method is to examine the two angles directly:



Geometry Expressions fails to simplify it by noticing that the numerator and denominator are identical.

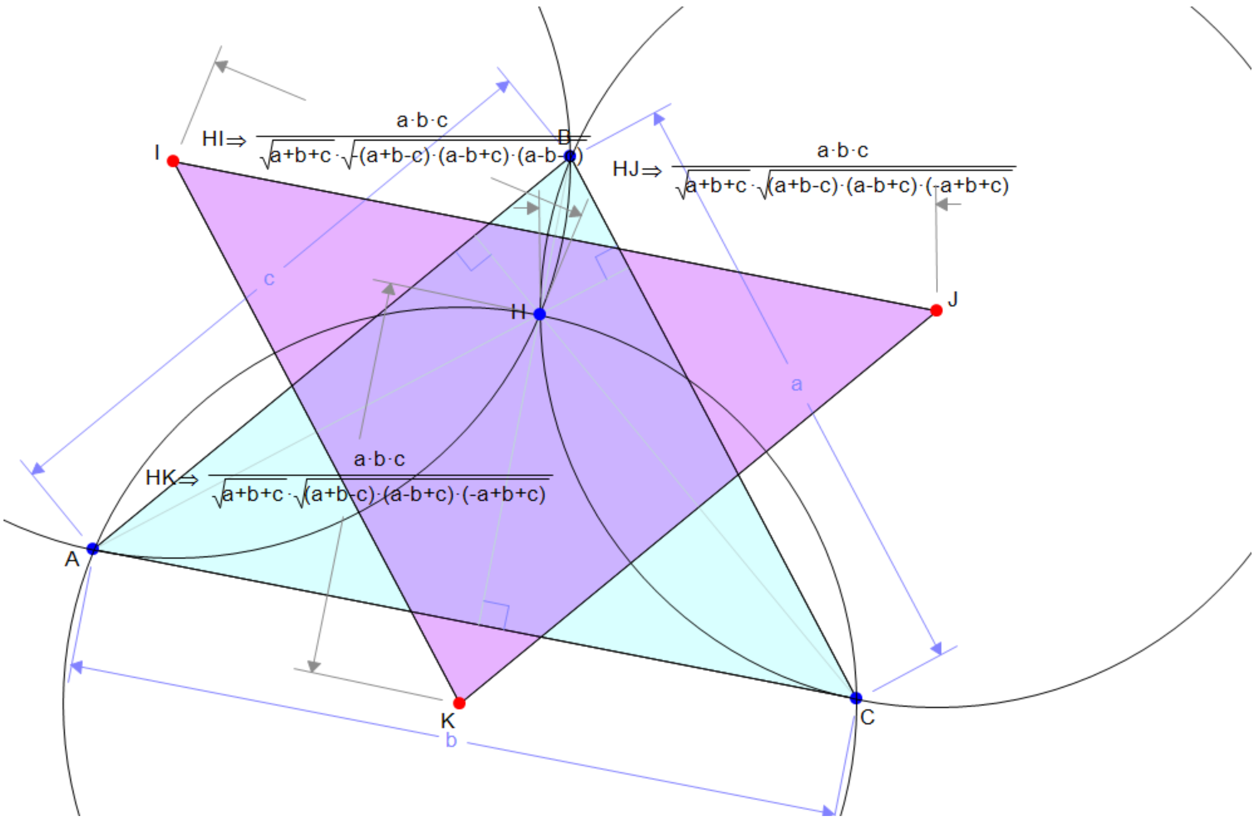
6.70

Let H be the orthocenter of triangle ABC. Then the circumcenters of the four triangles ABH, ACH, HBC form a triangle congruent to ABC. The sides are parallel.



6.71

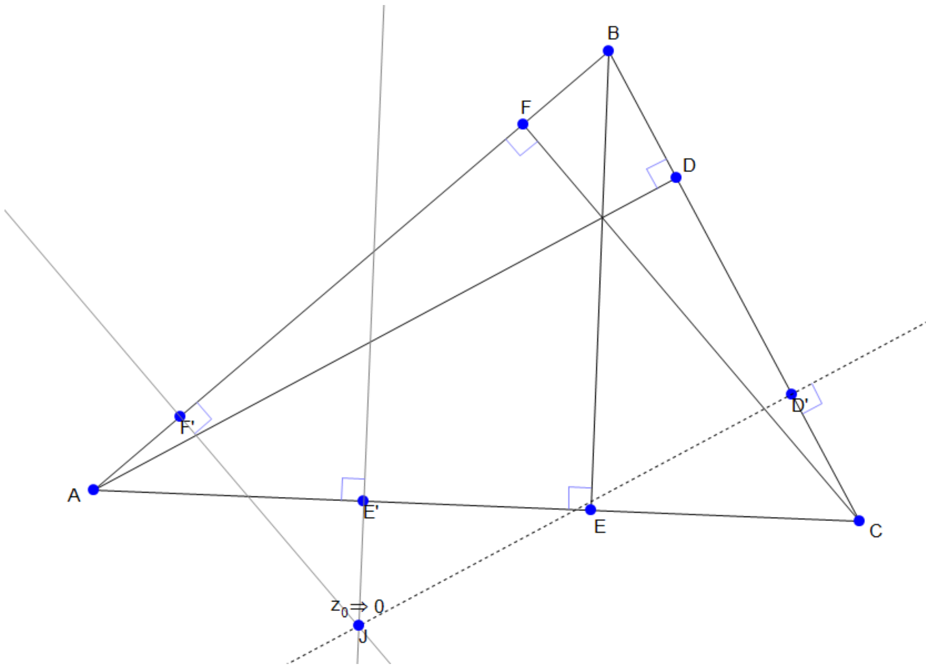
Continuing from Example 6.70, show that H is the circumcenter of IJK.



By inspection, the three lengths HI, HJ, HK are the same.

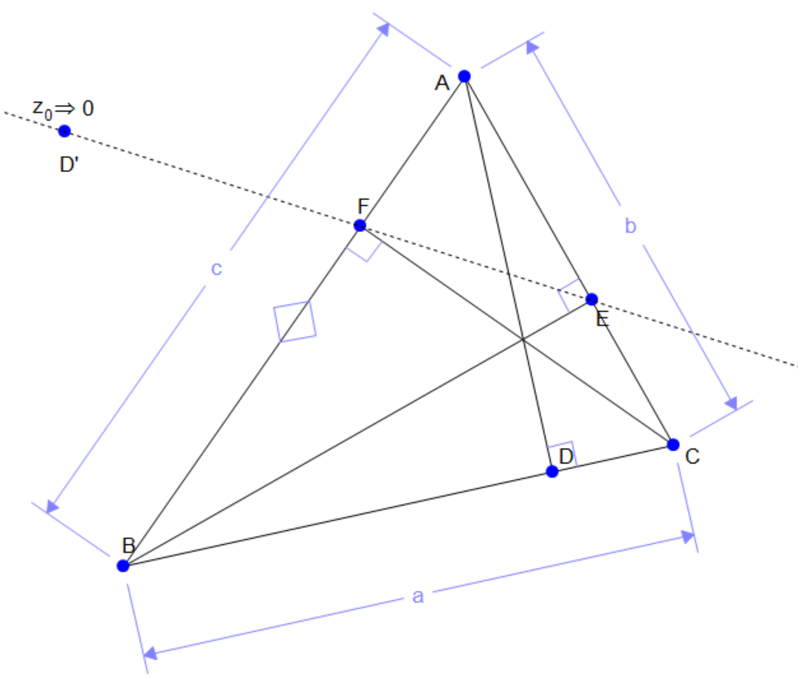
6.72

Show that the three perpendiculars to the sides of a triangle at the points isotomic to the foot of the respective altitudes are concurrent.



6.73

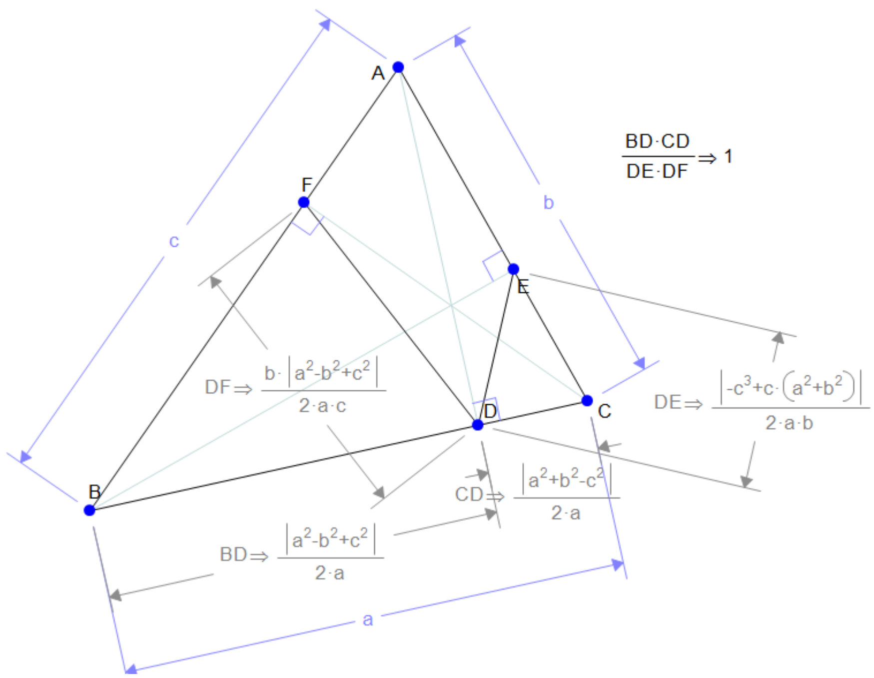
Show that the symmetries of the foot of the altitude to the base of the triangle with respect to the other two sides lie on the side of the orthic triangle relative to the base.



D’ is the image of D under reflection in AB. We show the distance from D’ to line EF is 0.

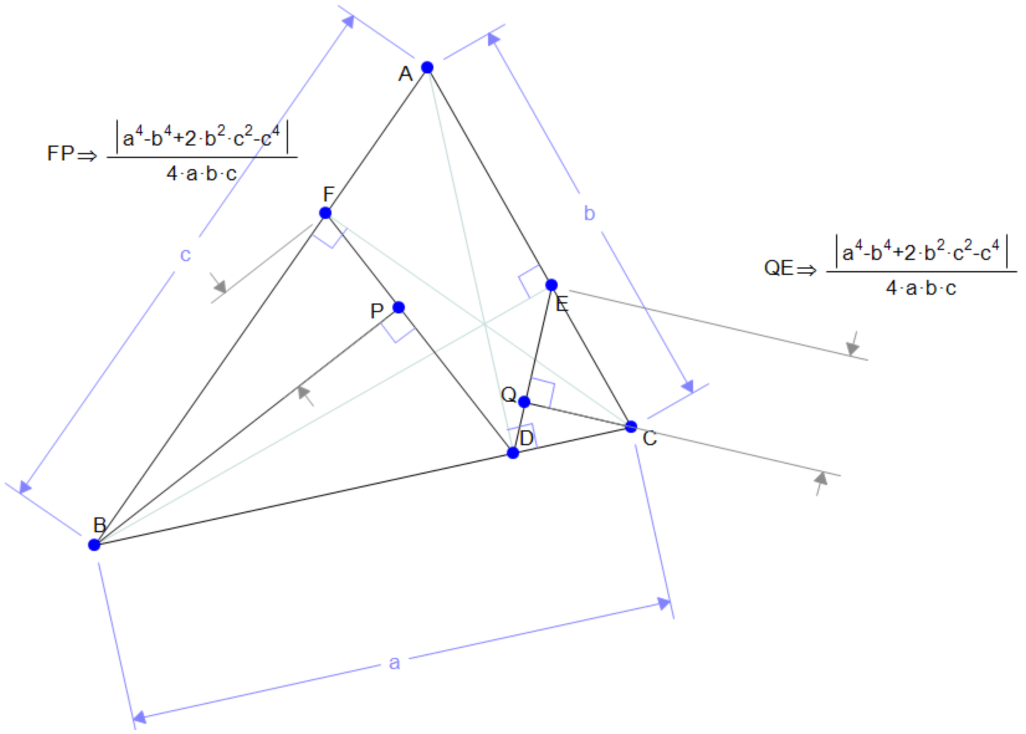
6.74

Show that the product of the segments into which a side of a triangle is divided by the corresponding vertex of the orthic triangle is equal to the product of the sides of the orthic triangle passing through the vertex considered.



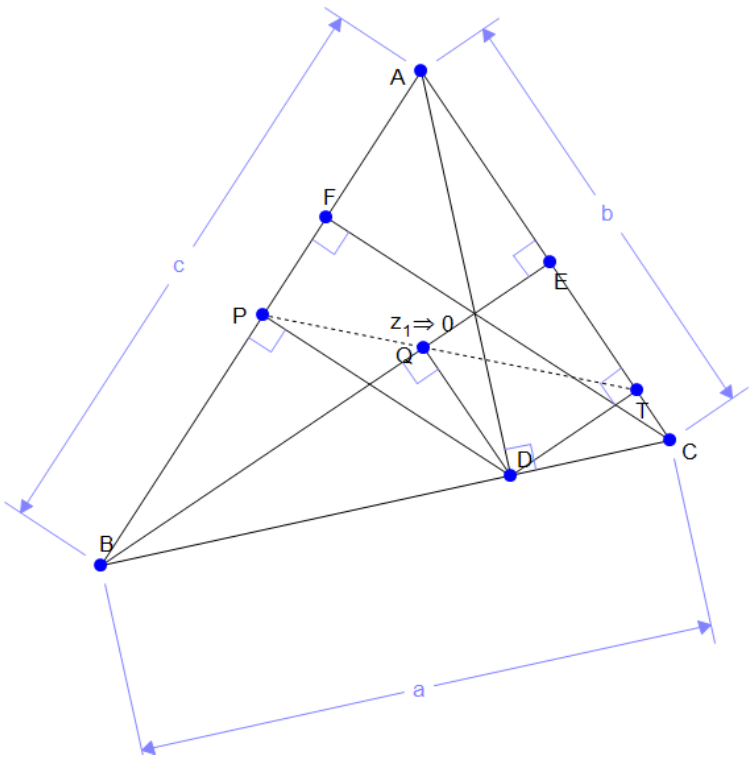
6.75

If P, Q are the feet of the perpendiculars from the vertices B, C of the triangle ABC on the sides DF, DE respectively, of the orthic triangle DEF, show that EQ=FP.



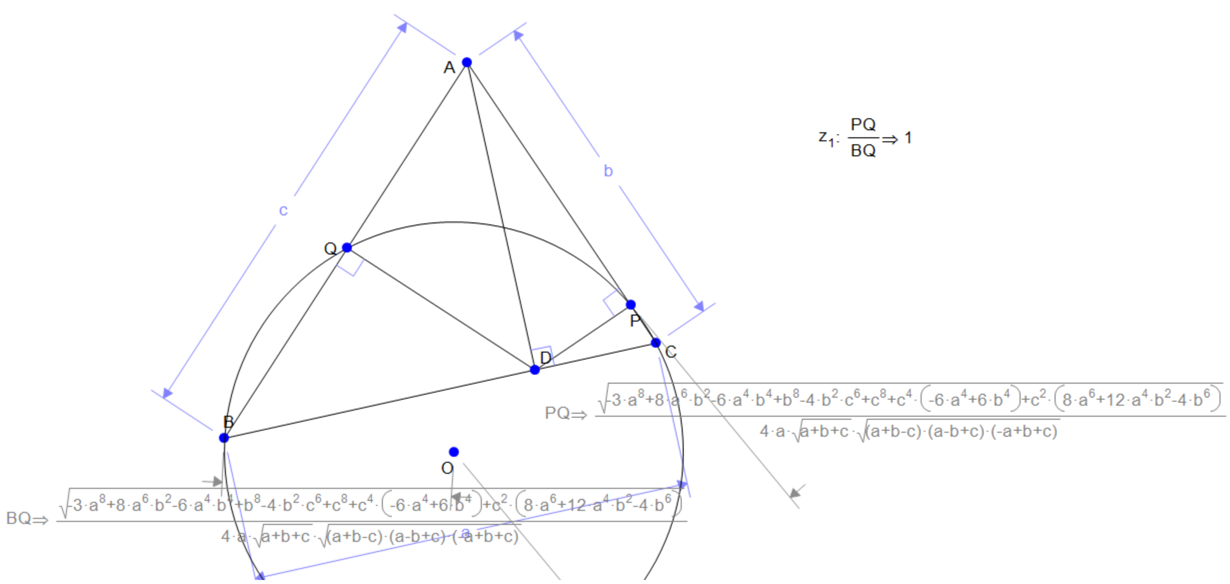
6.76

The four projections of the foot of the altitude on a side of a triangle upon the other two sides and the other two altitudes are collinear.



6.77

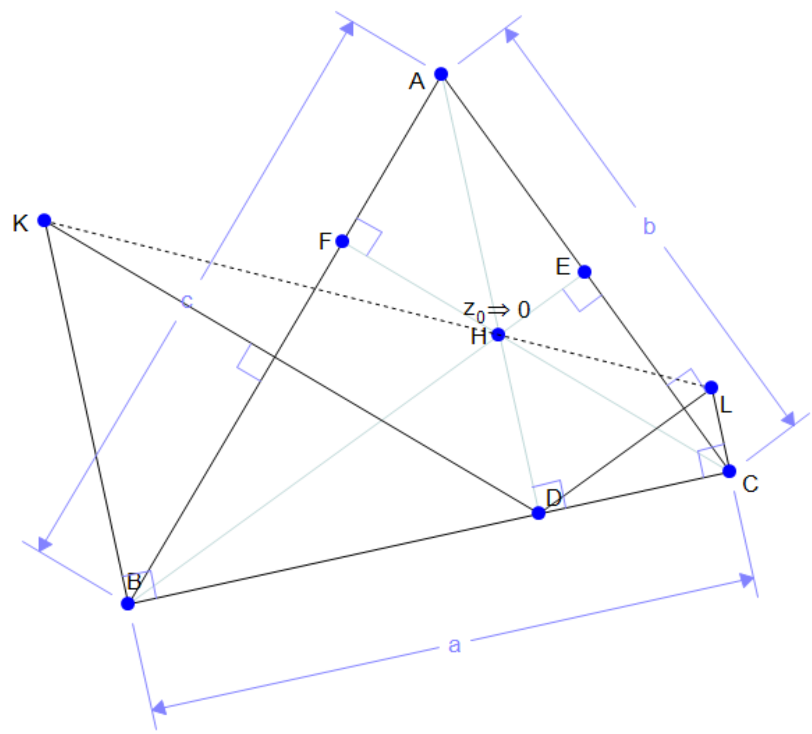
DP, DQ are perpendiculars from the foot D of the altitude AD of the triangle ABC on the sides AC, AB. Prove that the points B, C, P, Q are cyclic.



We put a circle through B, Q, C then show that P is the same distance from the center of this circle.

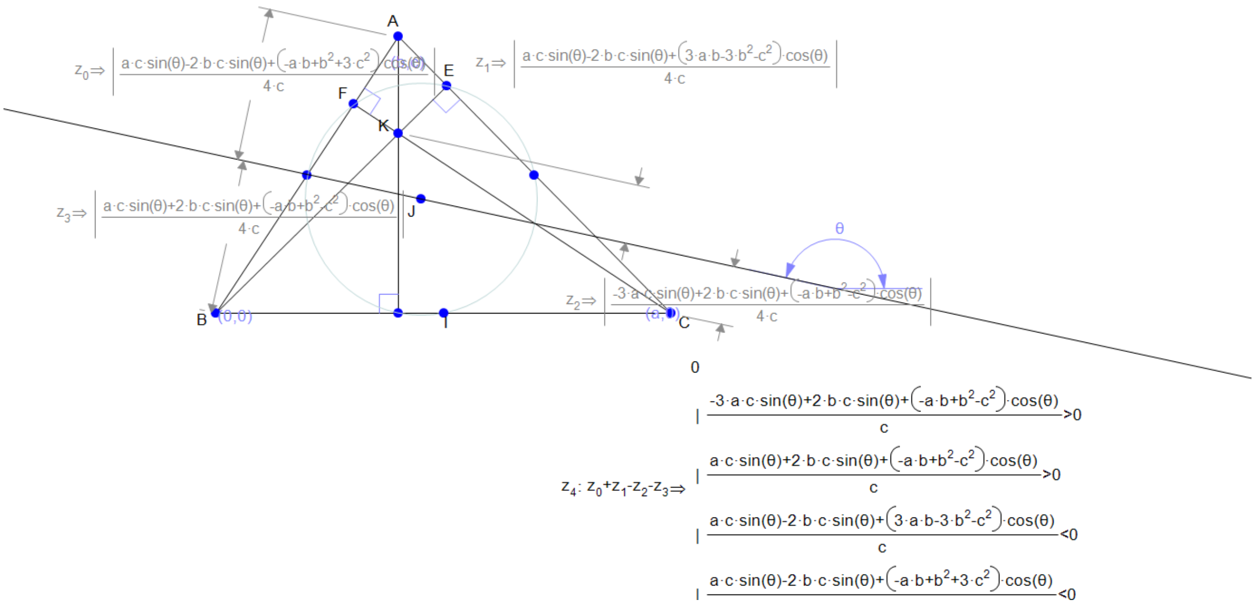
6.78

The perpendiculars DP, DQ dropped from the foot D of the altitude AD f the triangle ABC upon the sides AB, AC meet the perpendiculars BP, CQ erected to BC at B, C in the points P, Q respectively. Prove that the line PQ passes through the orthocenter H of ABC



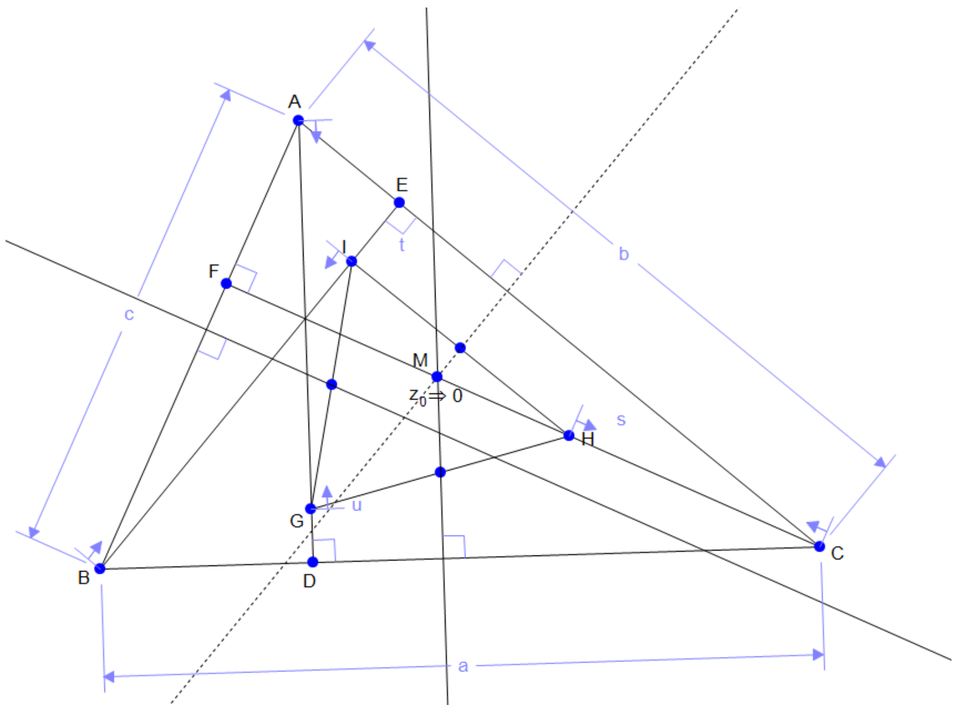
6.79

The algebraic sum of the distances of the points of an orthocentric group from any line passing through the nine-point center of the group is equal to zero.



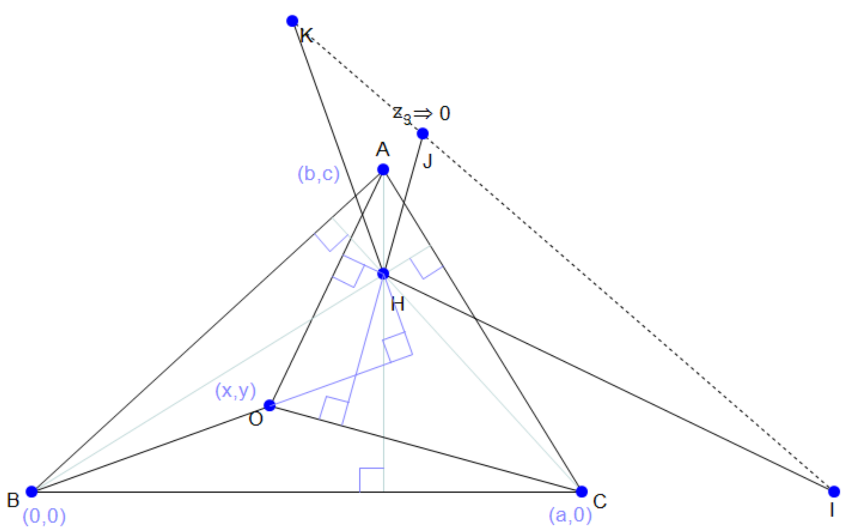
6.80

If through the midpoints of the sides of a triangle having its vertices on the altitudes of a given triangle, perpendiculars are dropped to the respective sides of the given triangle, show that the three perpendiculars are concurrent.



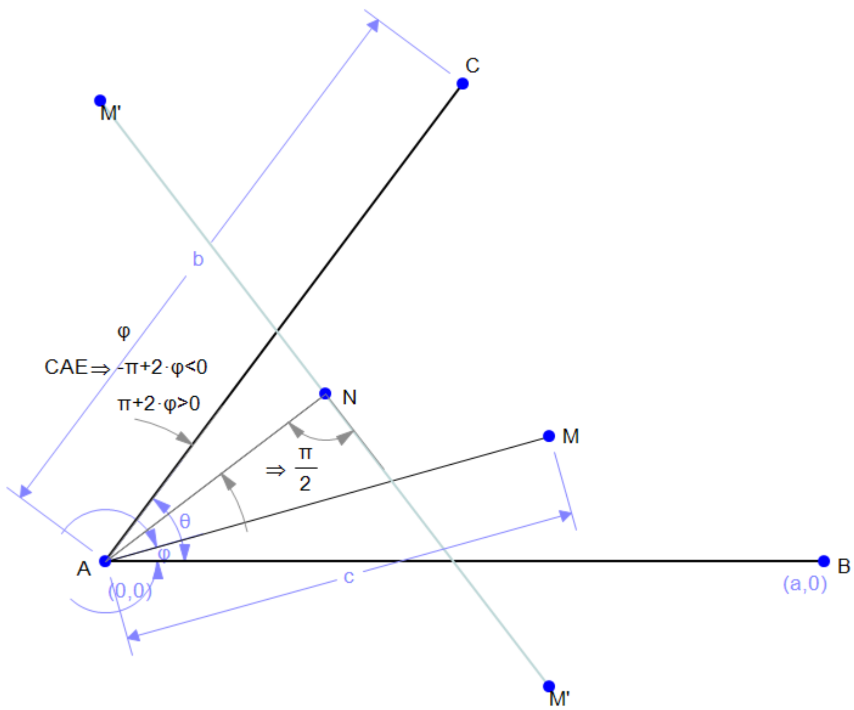
6.81

Show that the perpendiculars dropped from the orthocenter of a triangle upon the lines joining the vertices to a given point meet the respectively opposite sides of the triangle in three collinear points.



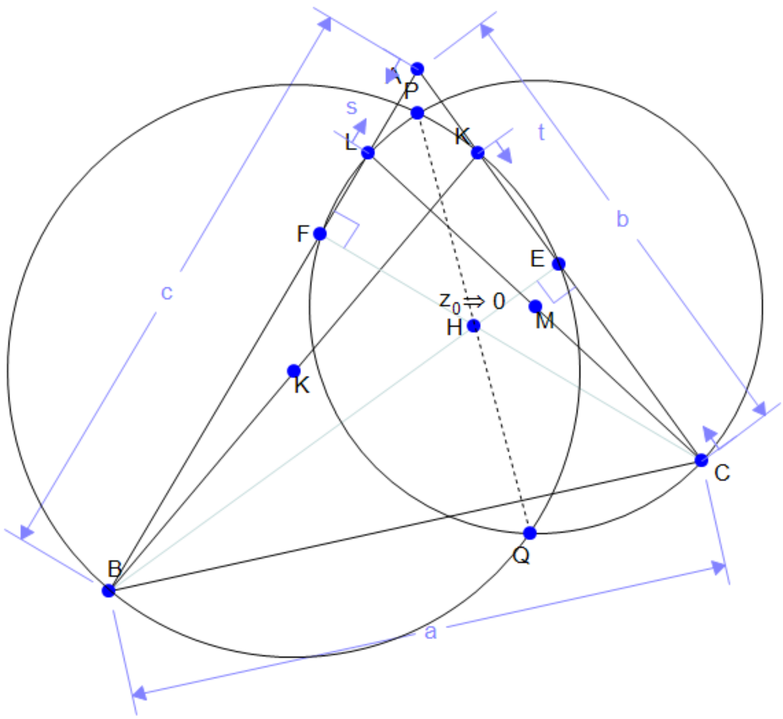
6.82

Show that the line joining a given point to the vertex of a given angle has for its isogonal line the mediator of the segment determined by the symmetries of the given point with respect to the sides of the angle.



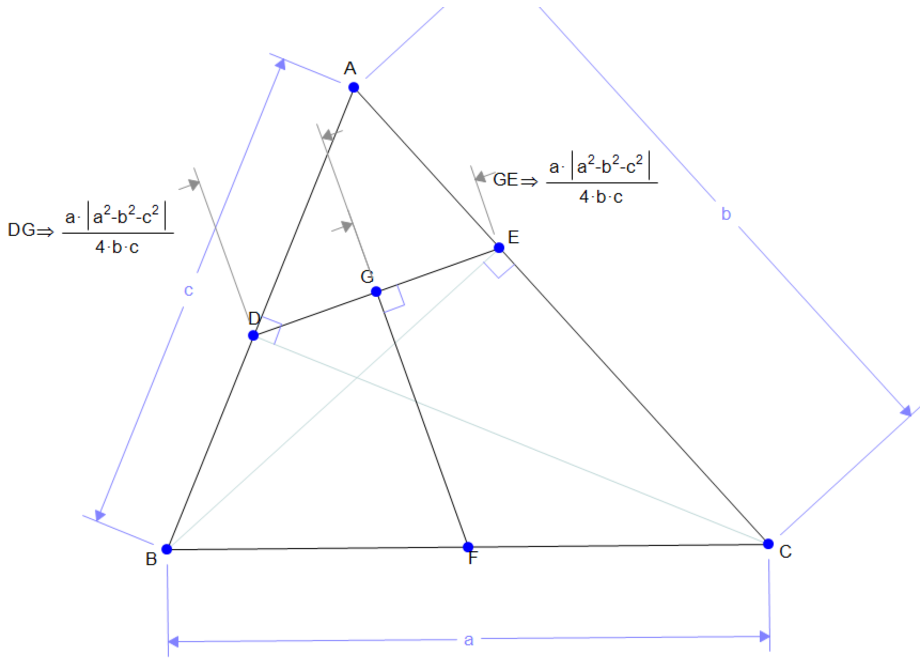
6.83

If circles are constructed on two Cevians as diameters, their radical axis passes through the orthocenter H of the triangle.



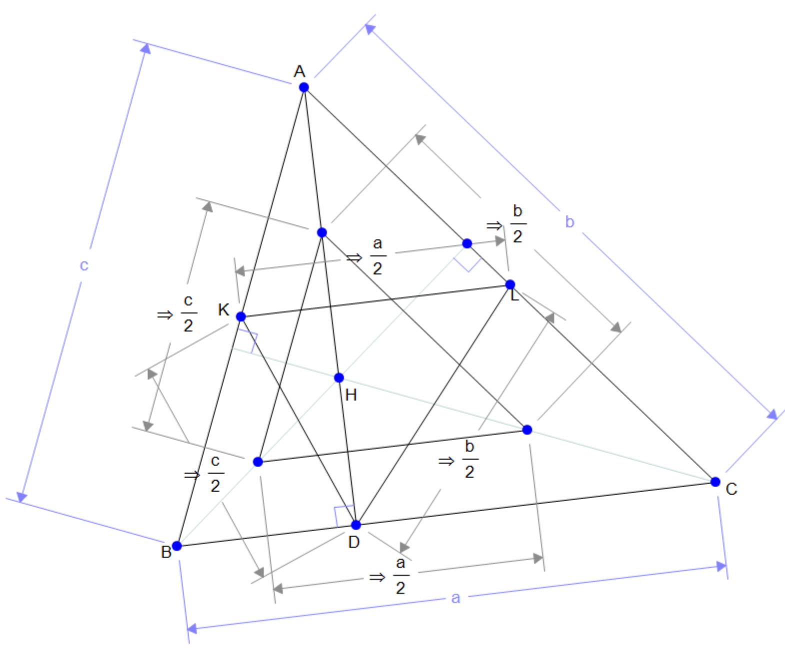
6.84

In triangle ABC, let F be the midpoint of the side BC, D and E the feet of the altitudes on AB, AC respectively. FG is perpendicular to DE at G. Show that G is the midpoint of DE.



6.85

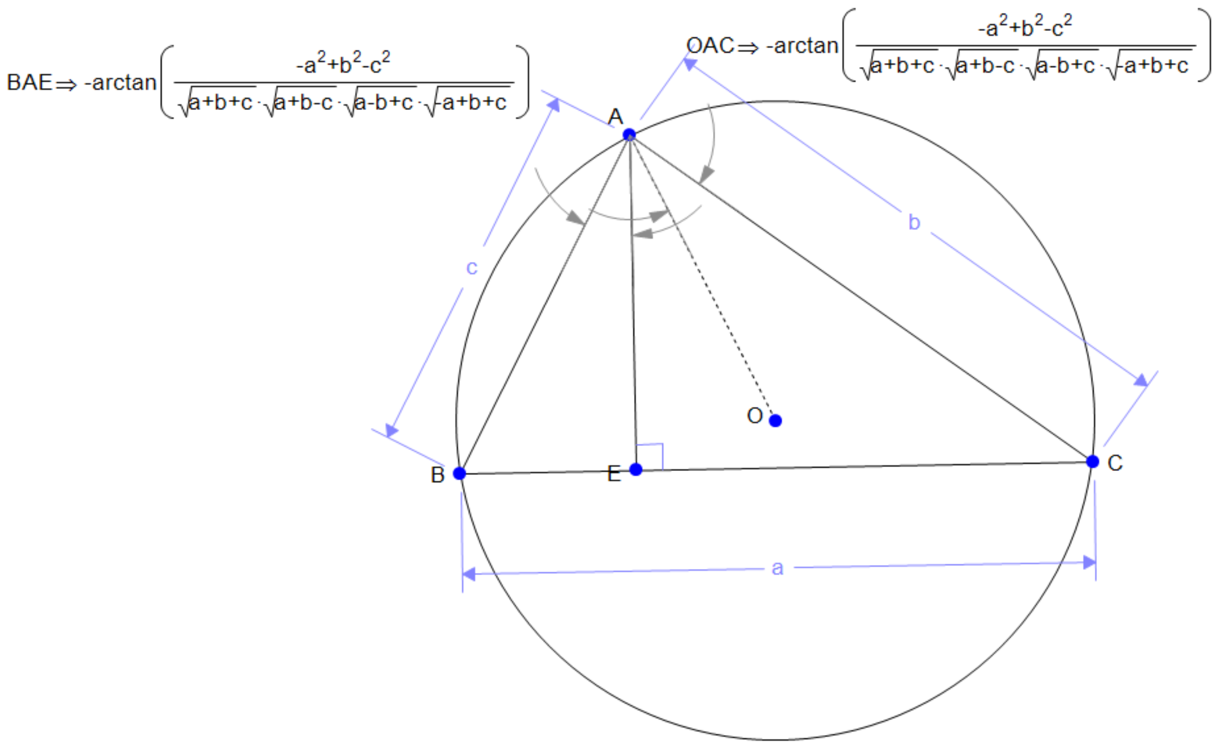
Let E and F be the midpoints of AC and AB, and D the foot of the altitude from A to BC. Show that the triangle DEF is congruent to the Euler triangle of ABC. (The Euler triangle has vertices on the altitudes of the triangle midway between the vertex and the orthocenter.)



2.3 The Circumcircle

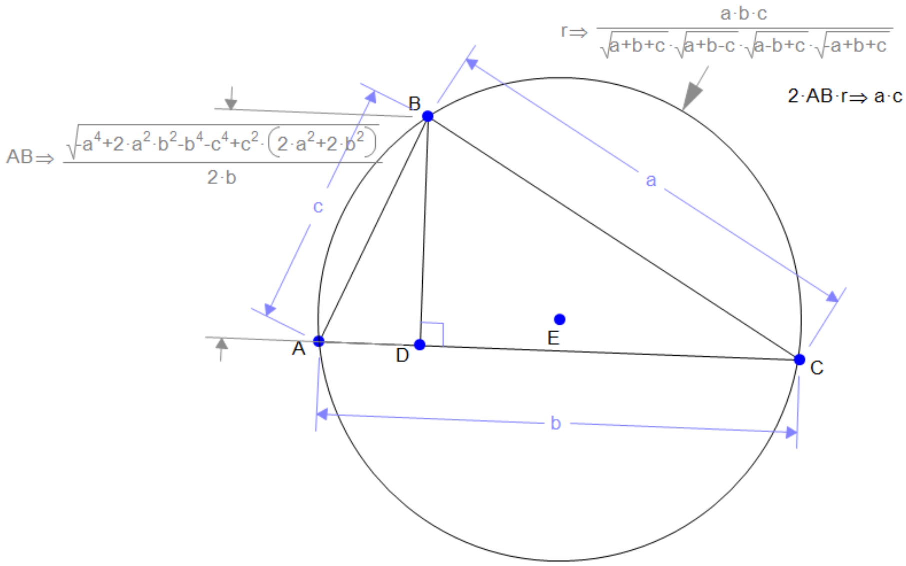
6.86

The angle between the circumdiameter and the altitude issued from the same vertex of a triangle is bisected by the bisector of the angle of the triangle at the vertex considered.



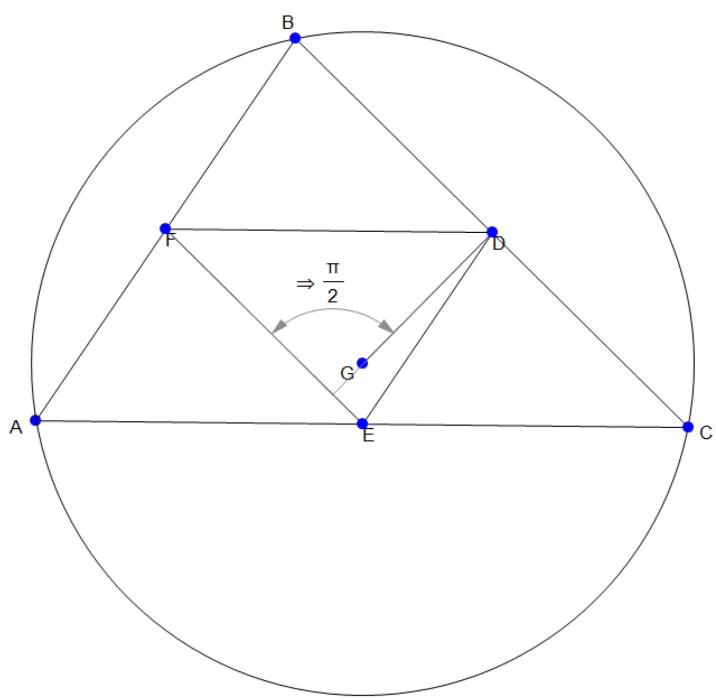
6.87

The product of two sides of a triangle is equal to the altitude to the third side multiplied by the circumdiameter.



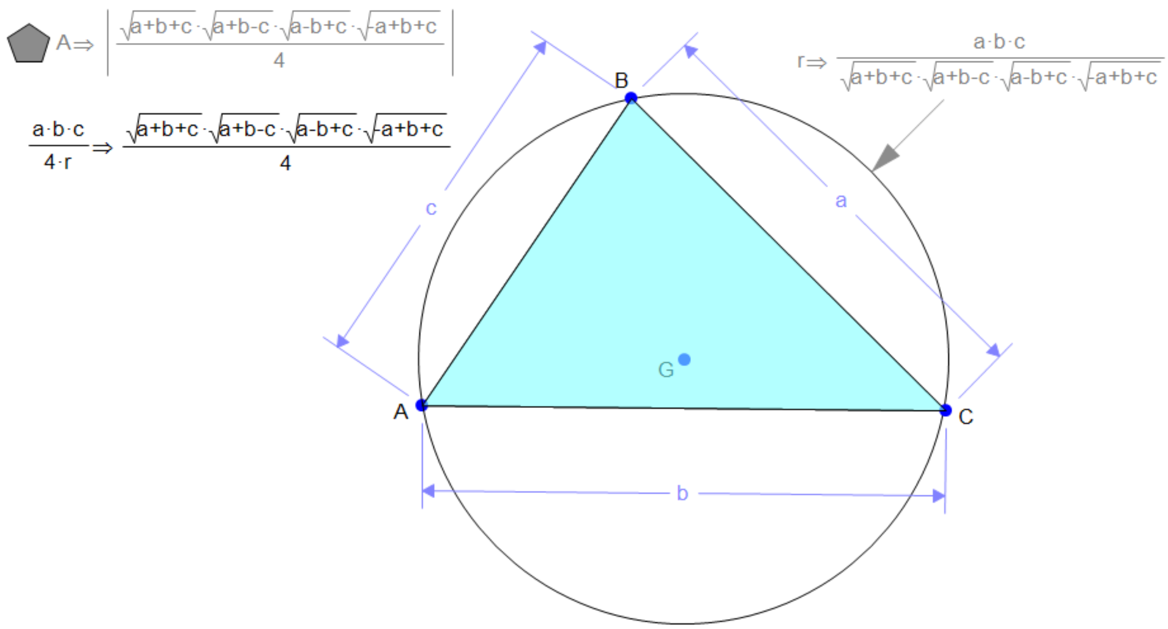
6.88

Prove that the circumcenter of a triangle is the orthocenter of its medial triangle



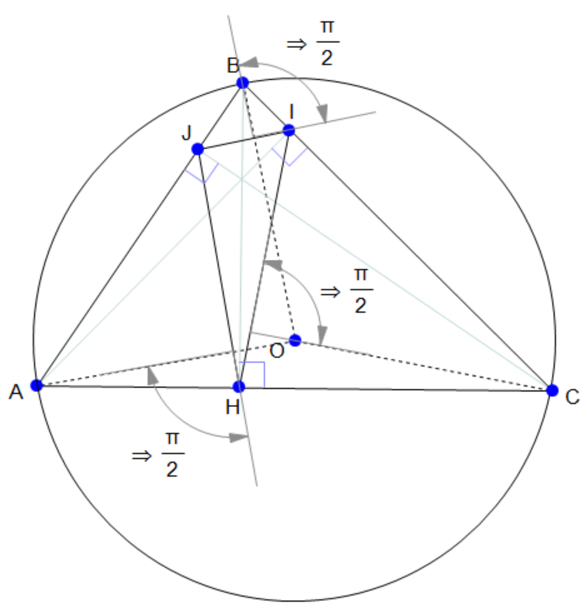
6.89

The area of a triangle is equal to the product of its three sides divided by the double circumdiameter of the triangle



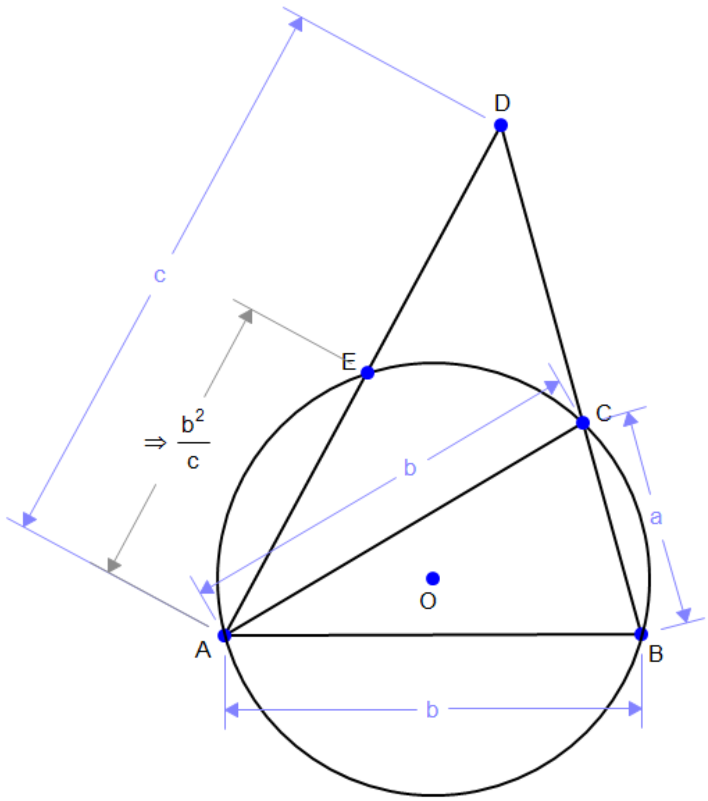
6.90

The radii of the circumcircle passing through the vertices of a triangle are perpendicular to the corresponding sides of the orthic triangle



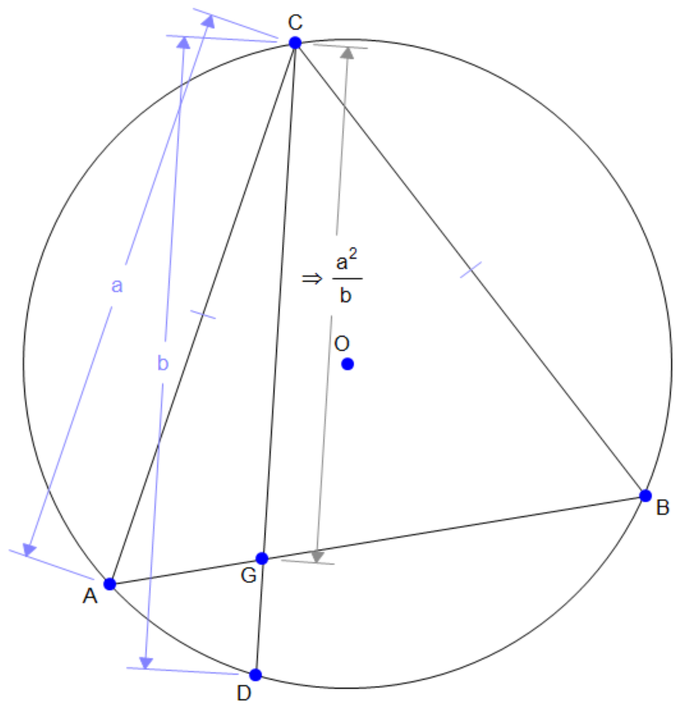
6.91

Let ABC be a triangle with AC=AB. D is a point on BC. Line AD meets the circumcircle of ABC at E. Show that AB^2=AD.AE



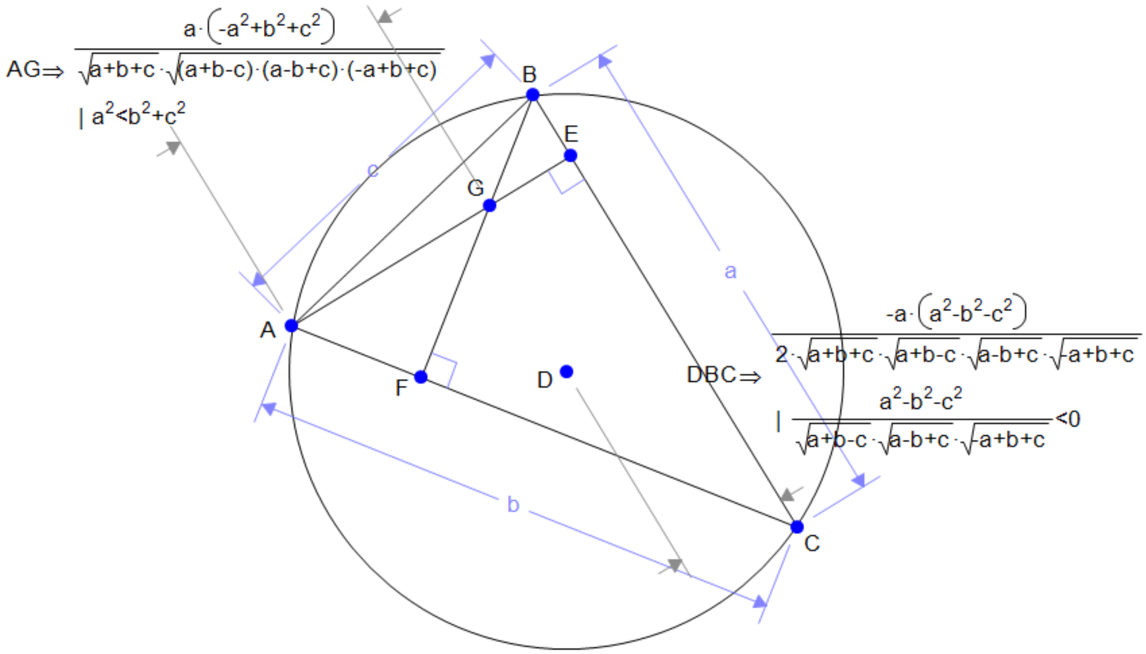
6.92

Let C be the midpoint of the arc AB of circle (O). D is a point on the circle. E is the intersection of AB and CD. Show that CA^2=CE.CD.



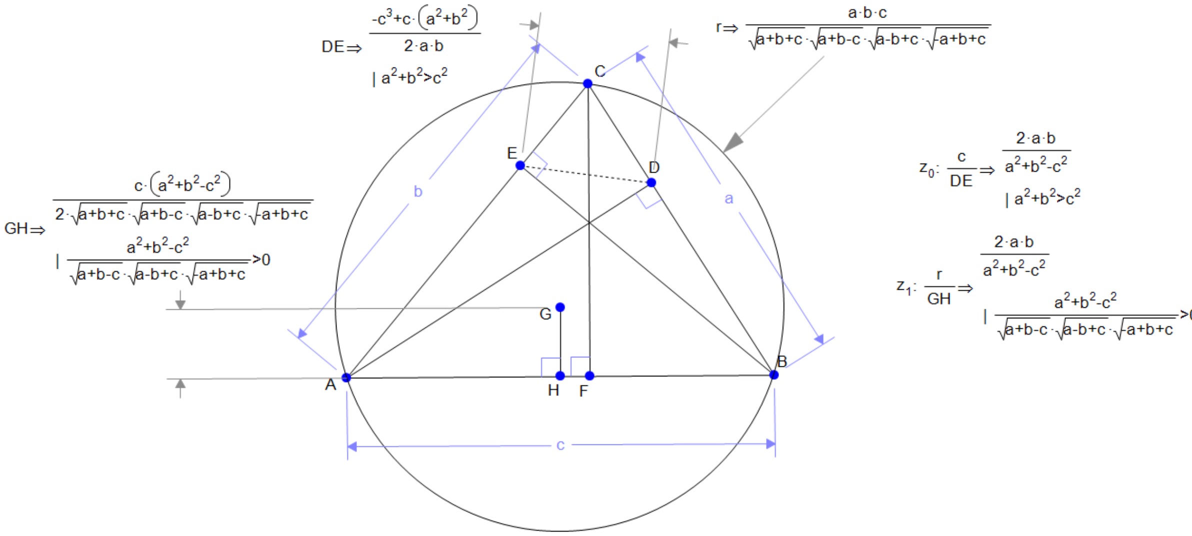
6.93

The distance of a side of a triangle from the circumcenter is equal to half the distance of the opposite vertex from the orthocenter.



6.94

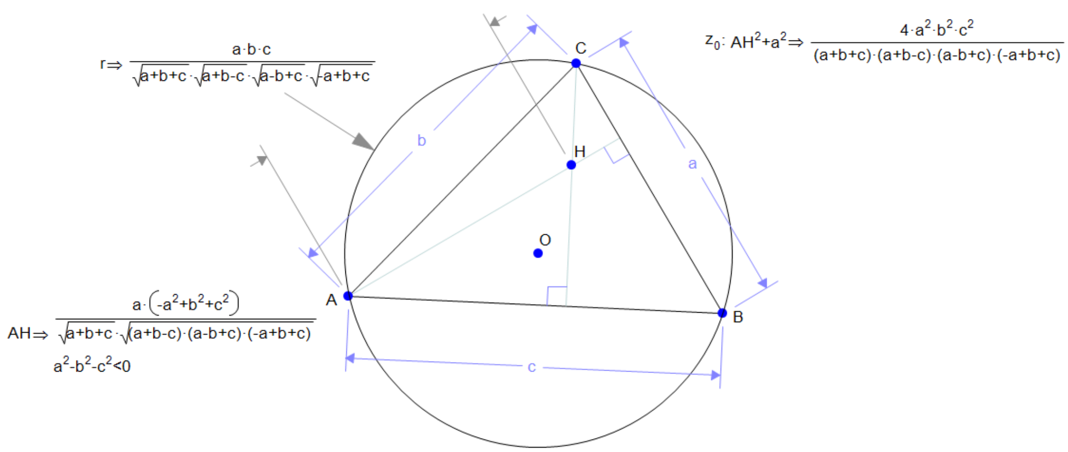
The ratio of a side of a triangle to the corresponding side of the orthic triangle is equal to the ratio of the circumference to the distance of the side considered from the circumcenter.



6.95

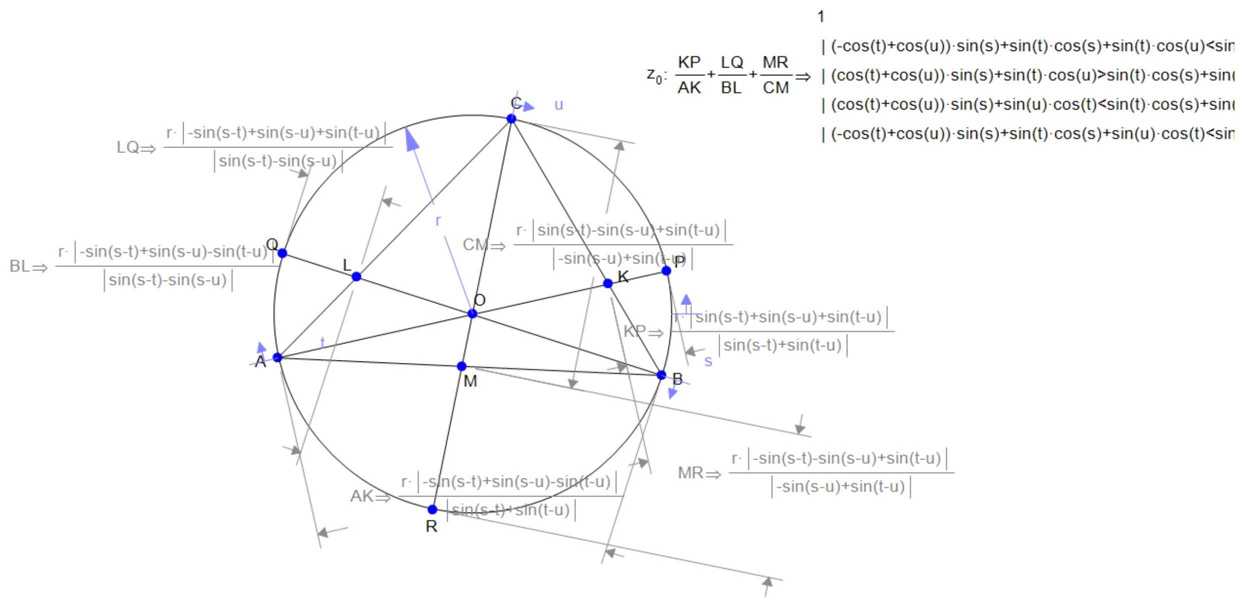
H is the orthocenter of triangle ABC and O is the circumcenter.

$AH^2+BC^2=4OA^2$



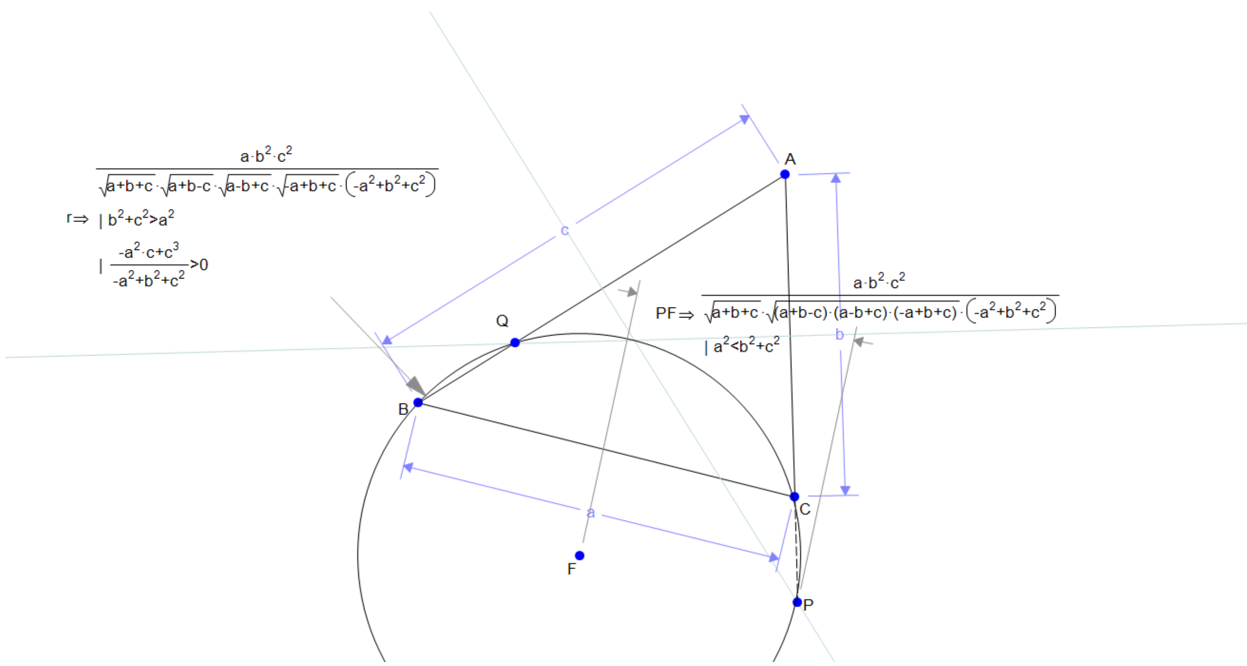
6.96

The circumdiameters AP, BQ, CR of a triangle ABC meet the sides BC, CA, AB in the points K, L, M. Show that $\frac{KP}{AK} + \frac{LQ}{BL} + \frac{MR}{CM} = 1$



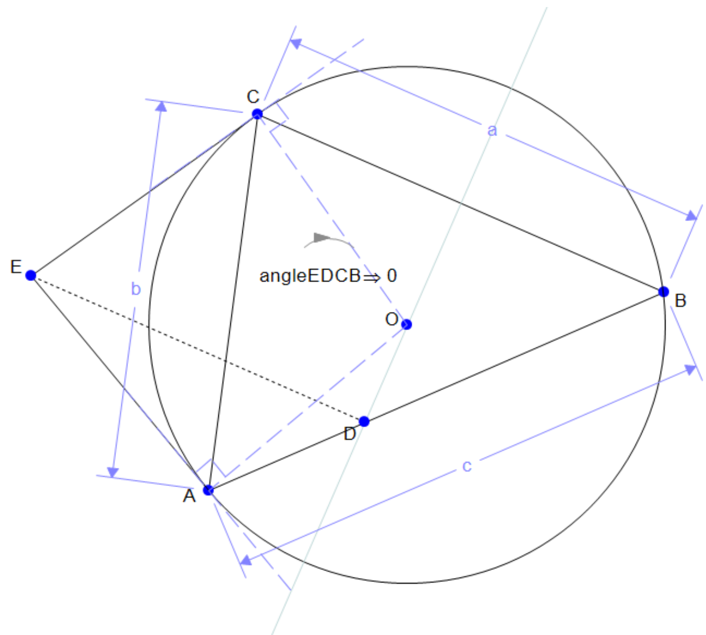
6.97

The mediators of the sides AC, AB of the triangle ABC meet the sides AB, AC in P and Q. Prove that the points B, C, P, Q lie on a circle.



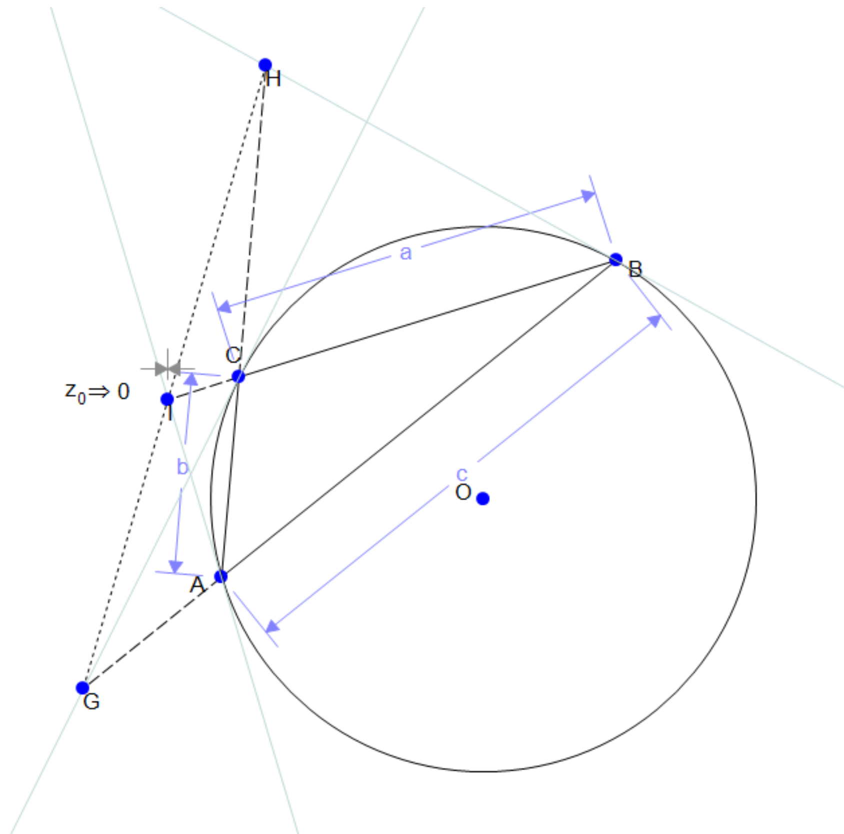
6.98

The two tangents to the circumcircle of ABC at A and C meet at E. The mediator of BC meets AB at D. Show that DE is parallel to BC.



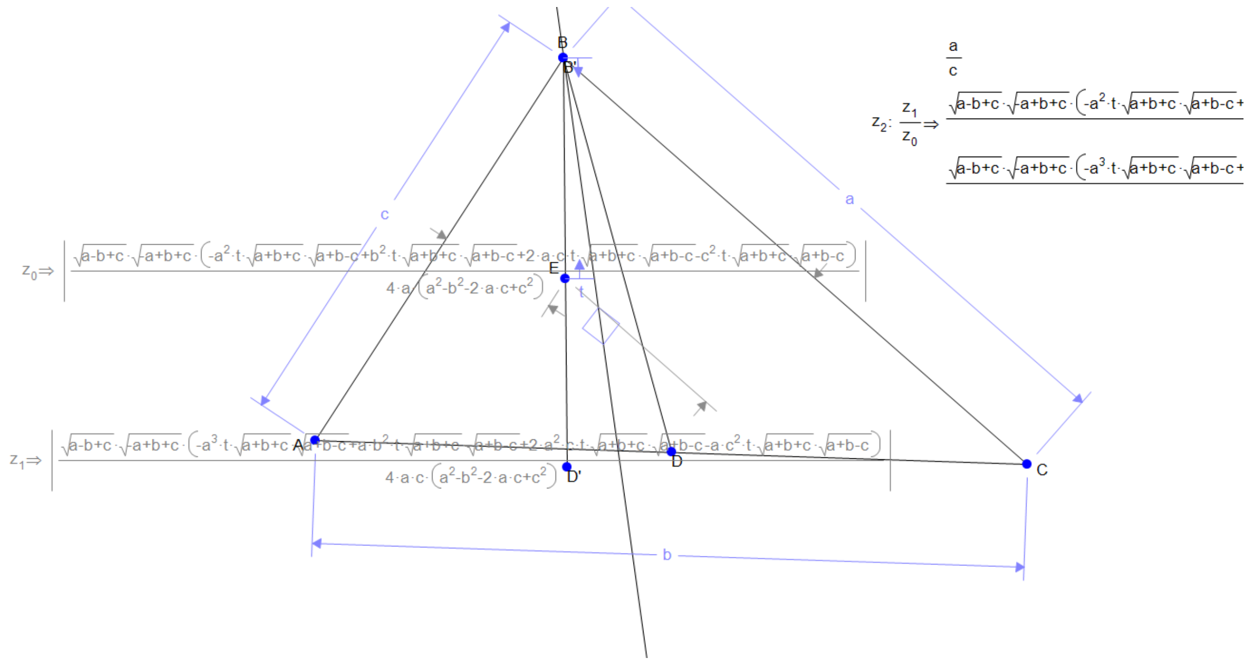
6.99

The lines tangent to the circumcircle of a triangle at the vertices meet opposite sides in three collinear points (the Lemoine axis of the triangle).



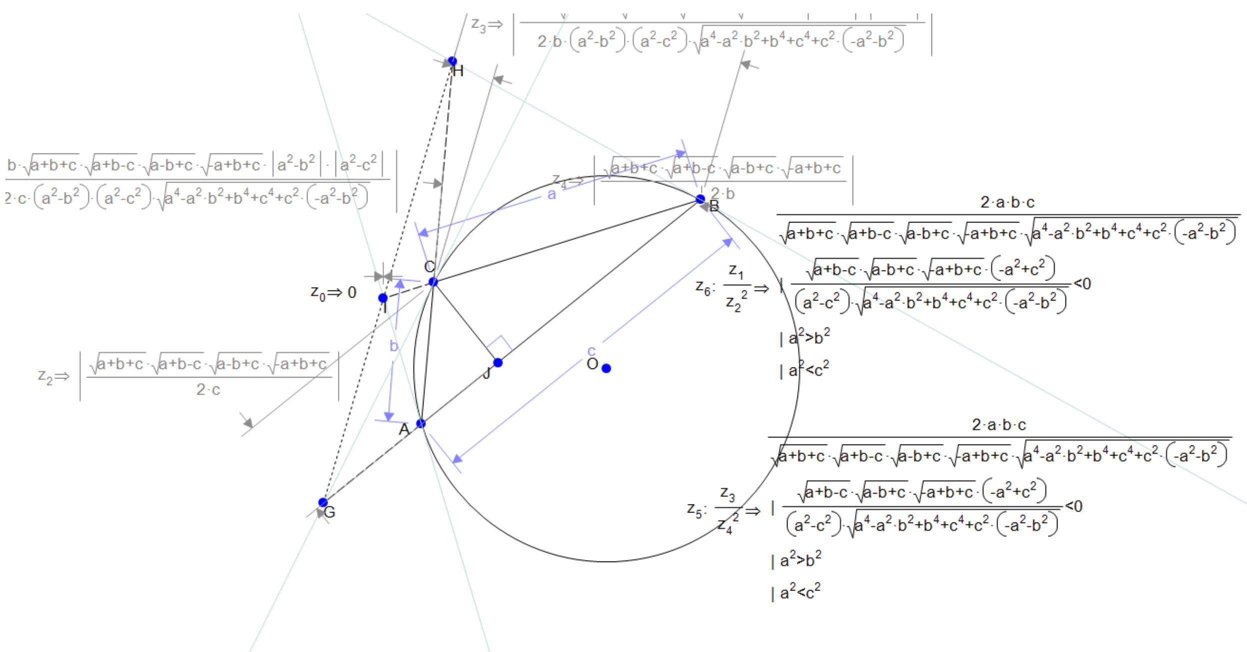
6.100

The distances from a point on the symmedian of a triangle to the two including sides are proportional to those sides.



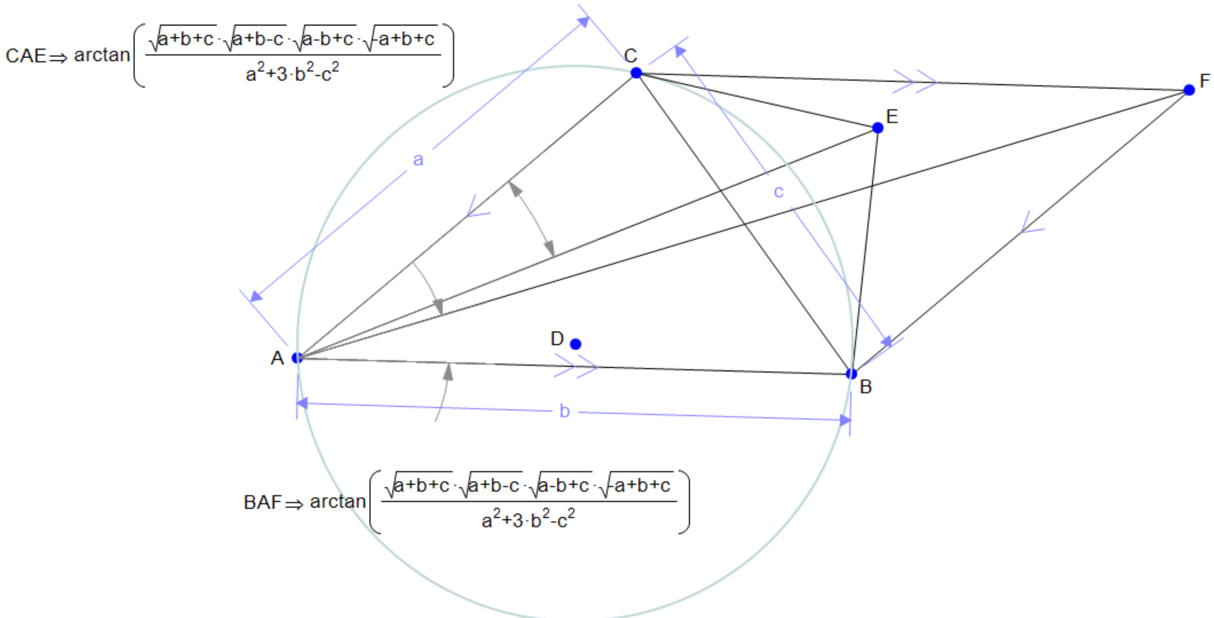
6.101

The distances of the vertices of a triangle from the Lemoine axis are proportional to the squares of the respective altitudes.



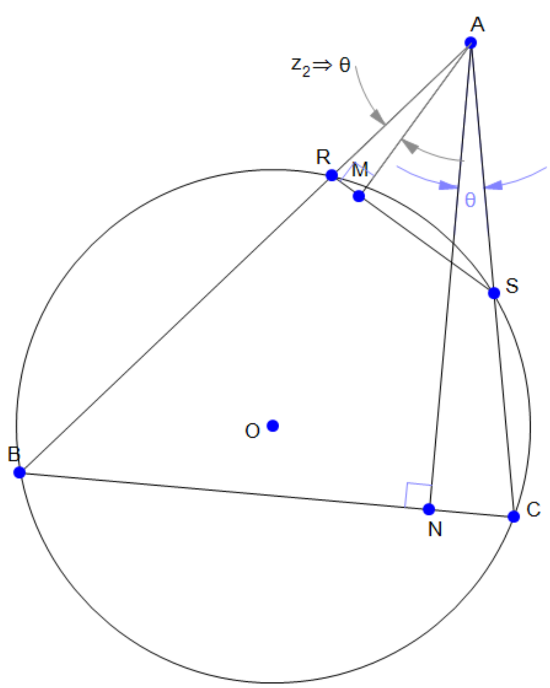
6.102

Show that the vertices of the tangential triangle of ABC are the isogonal conjugates of the anticomplementary triangle of ABC



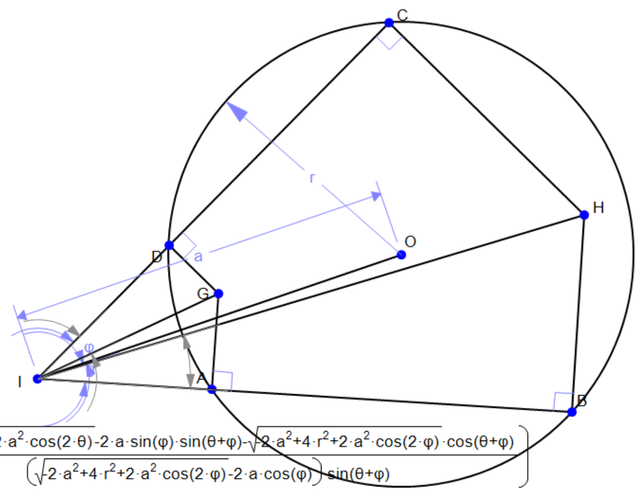
6.103

If two lines are antiparallel with respect to an angle, the perpendiculars dropped upon them from the vertex are isogonal in the angle considered.



6.104

Show that the four perpendiculars to the sides of an angle at four cyclic points form a parallelogram whose opposite vertices lie on isogonal conjugate lines with respect to the angle



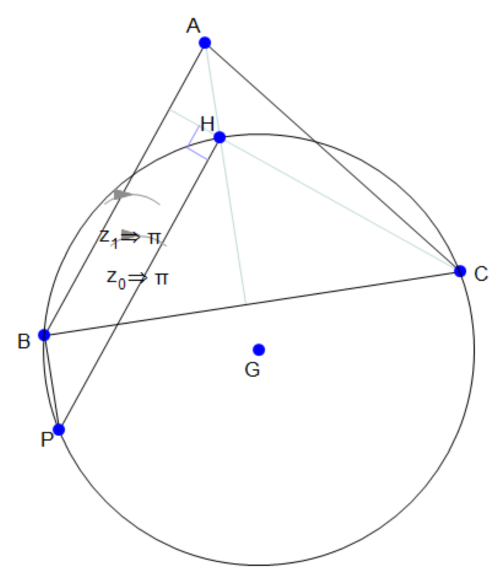
$$\Rightarrow \arctan\left(\frac{\sqrt{2\cdot a^2+4\cdot r^2+2\cdot a^2\cdot\cos(2\cdot\theta)}-2\cdot a\cdot\sin(\varphi)\cdot\sin(\theta+\varphi)-\sqrt{2\cdot a^2+4\cdot r^2+2\cdot a^2\cdot\cos(2\cdot\varphi)}\cdot\cos(\theta+\varphi)}{\left(\sqrt{2\cdot a^2+4\cdot r^2+2\cdot a^2\cdot\cos(2\cdot\varphi)}-2\cdot a\cdot\cos(\varphi)\right)\sin(\theta+\varphi)}\right)$$

$$\Rightarrow -\arctan\left(\frac{-\sqrt{2\cdot a^2+4\cdot r^2+2\cdot a^2\cdot\cos(2\cdot\varphi)}-2\cdot a\cdot\sin(\theta)\cdot\sin(\theta+\varphi)+\sqrt{2\cdot a^2+4\cdot r^2+2\cdot a^2\cdot\cos(2\cdot\theta)}\cdot\cos(\theta+\varphi)}{\left(\sqrt{2\cdot a^2+4\cdot r^2+2\cdot a^2\cdot\cos(2\cdot\theta)}+2\cdot a\cdot\cos(\theta)\right)\cdot\sin(\theta+\varphi)}\right)$$

These two are not self-evidently equal and require some significant Maple processing to show that they are.

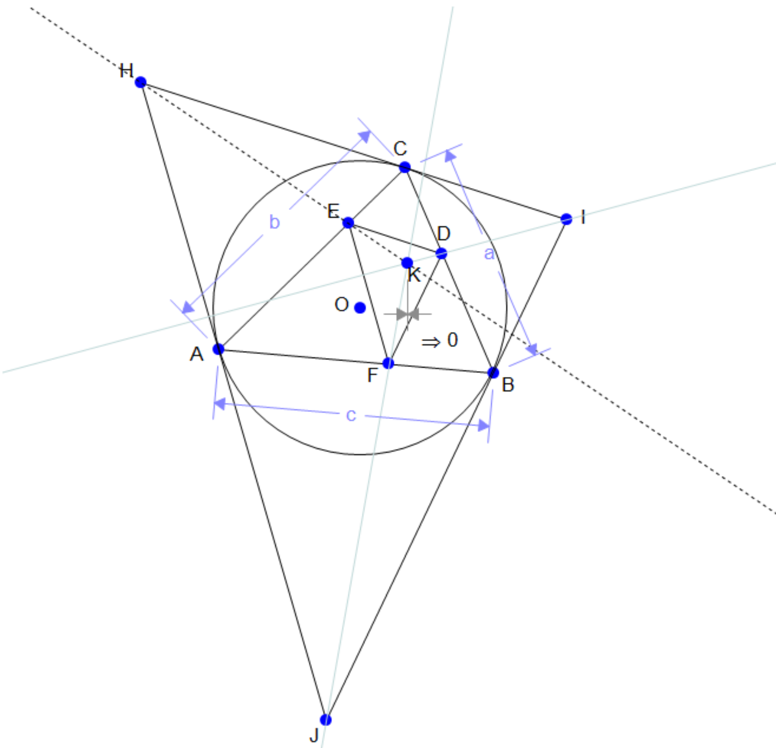
6.105

The perpendicular at the orthocenter H to the altitude HC of the triangle ABC meets the circumcircle of HBC in P. Show that ABPH is a parallelogram.



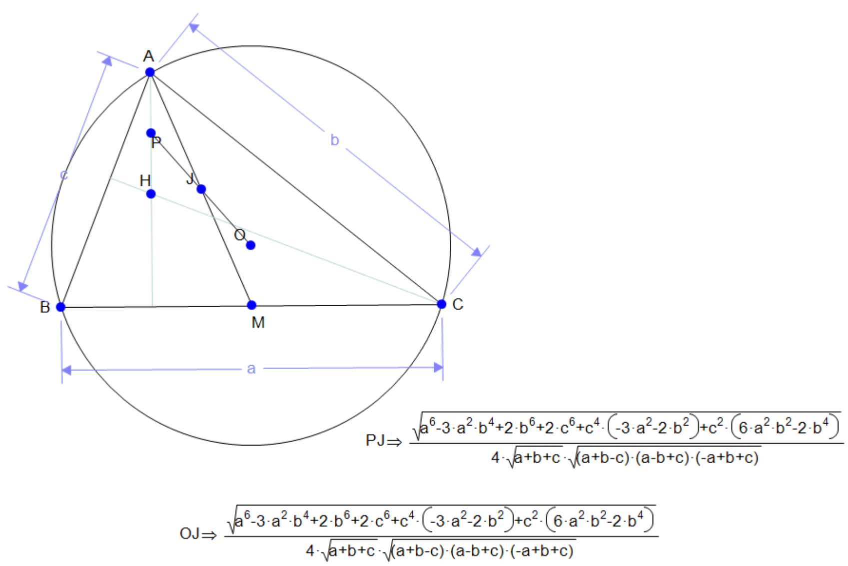
6.106

The tangential and orthic triangles of a given triangle are homothetic



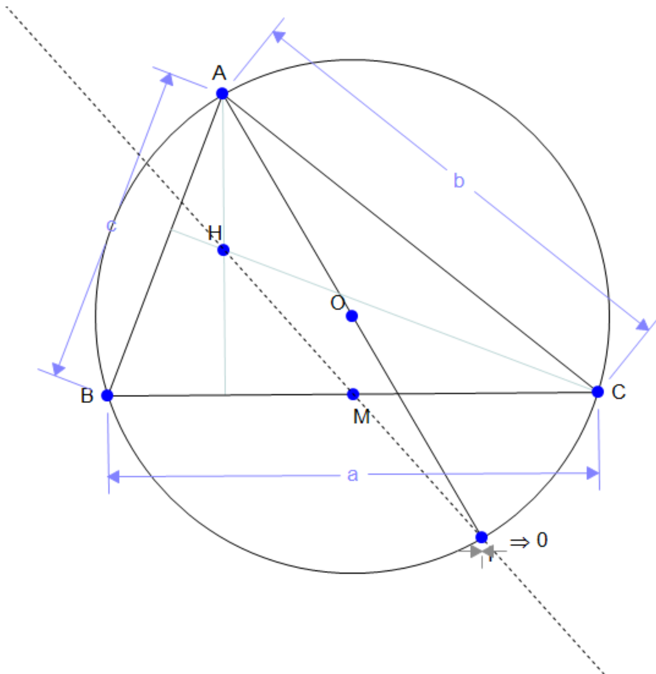
6.107

Let P be the midpoint of AH. Show that the segment OP is bisected by the median AM.



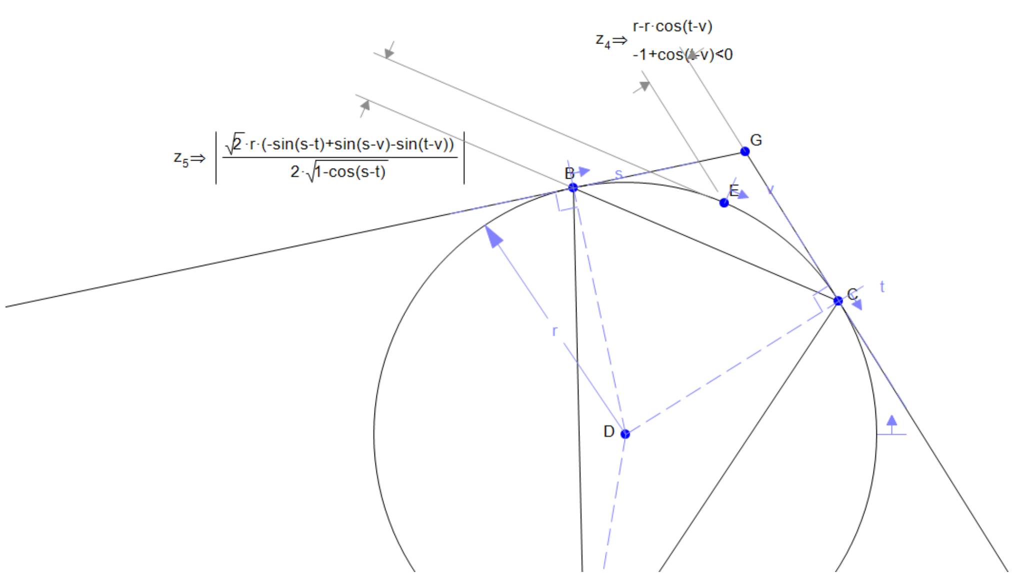
6.108

Prove that HM (see above) passes through the diametric opposite of A on the circumcircle



6.109

Show that the product of the distances of a point of the circumcircle of a triangle from the sides of a triangle is equal to the product of the distances of the same point from the sides of the tangential triangle.

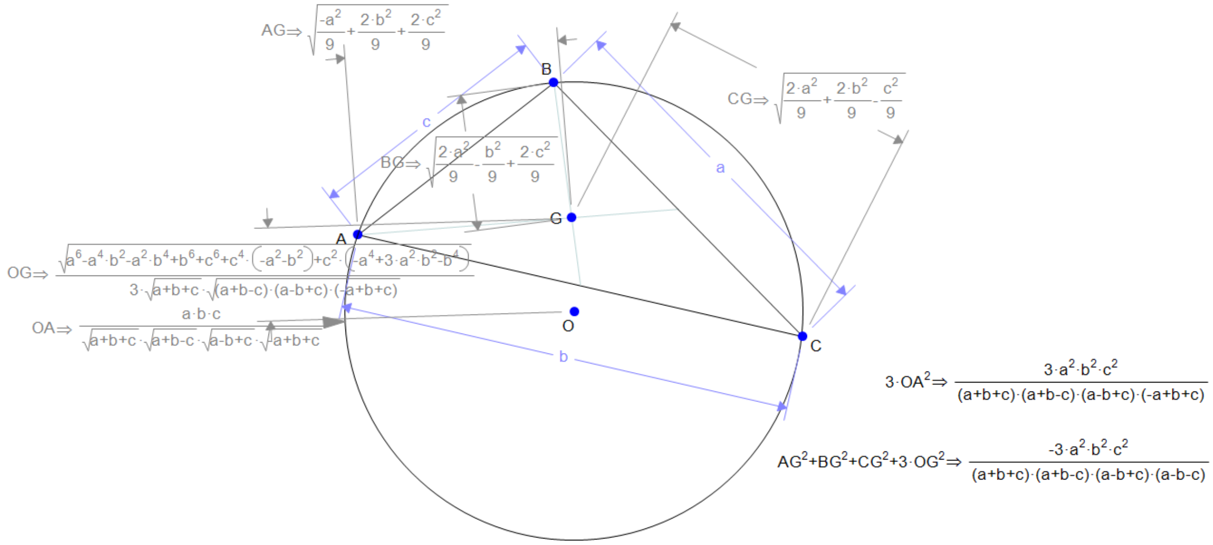


We fail on this one, note that the distance from E to the tangential triangle is nice and simple, but the distance to the internal triangle is not.

6.110

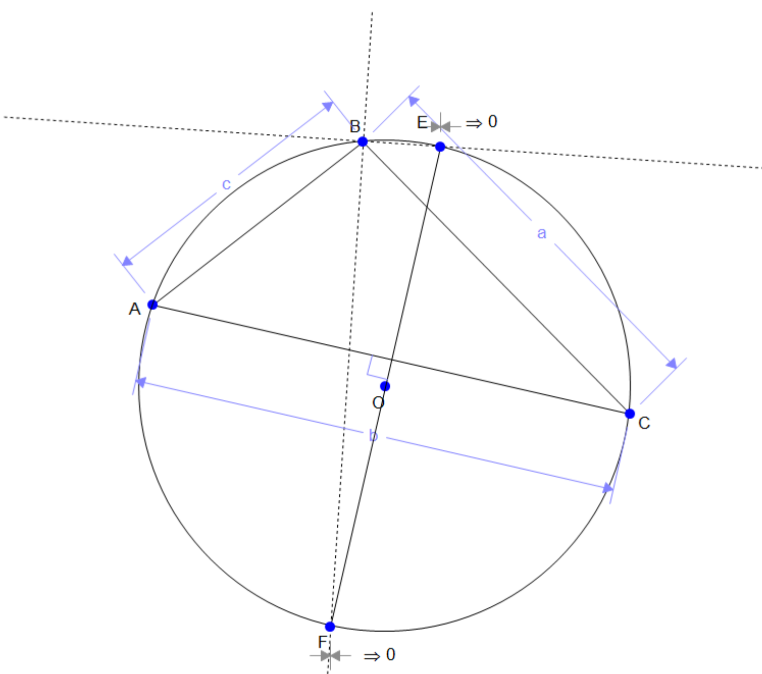
If O is the circumcenter of the triangle ABC and G is its centroid we have

$$3OA^2 = GA^2 + GB^2 + GC^2 + 3OG^2$$



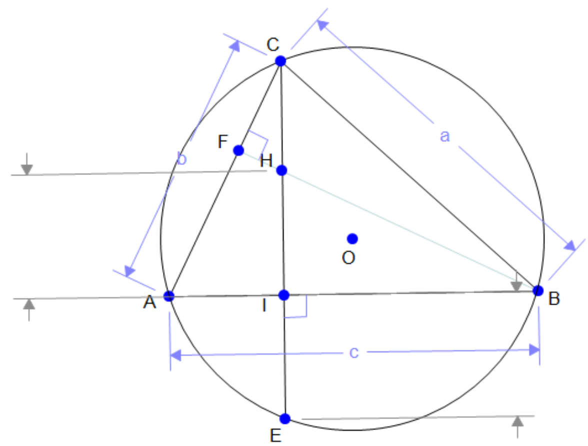
6.111

The internal and external bisectors of an angle of a triangle pass through the ends of the circumdiameter which is perpendicular to the side opposite the vertex considered



6.112

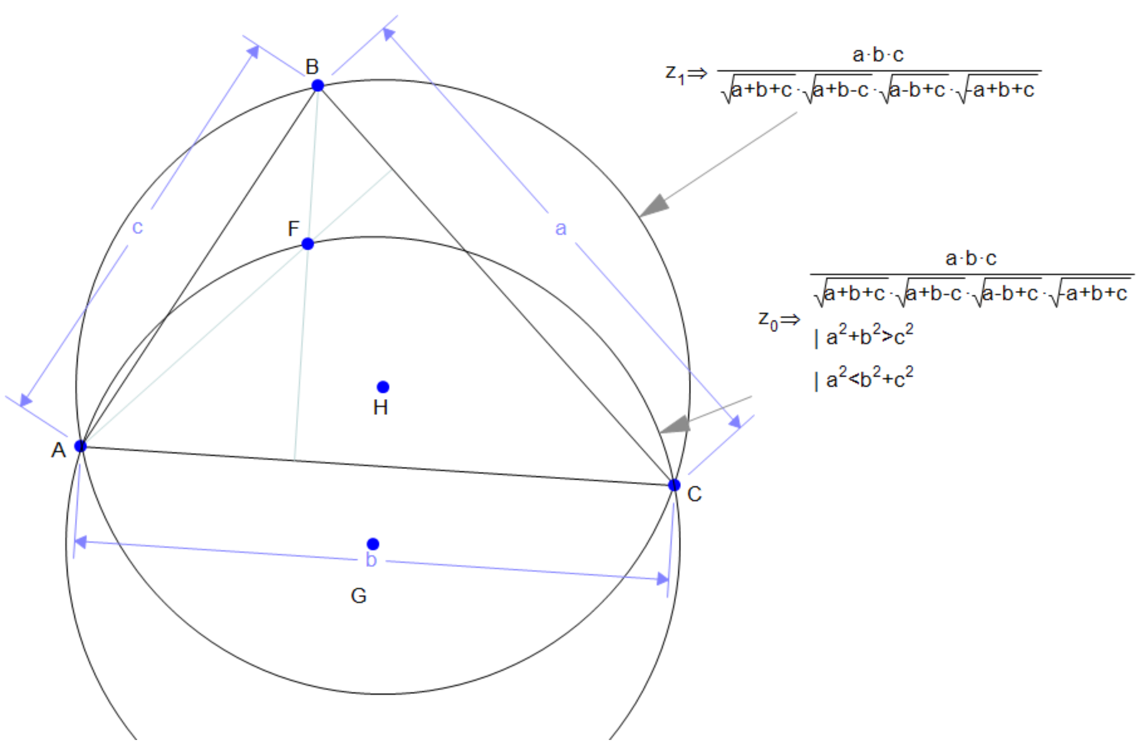
The segment of the altitude extended between the orthocenter and the second point of intersection with the circumcircle is bisected by the corresponding side of the triangle



$$HI \Rightarrow \left| \frac{-a^4+2\cdot a^2\cdot b^2-b^4+c^4}{2\cdot c\cdot\sqrt{a+b+c}\cdot\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}} \right| \qquad EI \Rightarrow \left| \frac{-a^4+2\cdot a^2\cdot b^2-b^4+c^4}{2\cdot c\cdot\sqrt{a+b+c}\cdot\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}} \right|$$

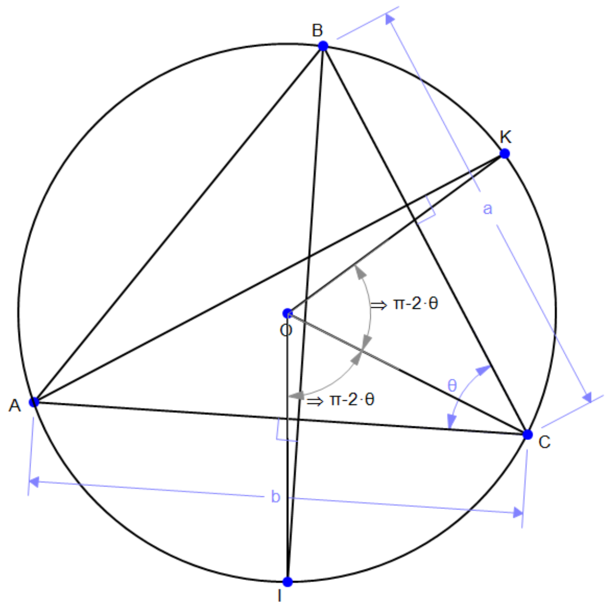
6.113

The circumcircle of the triangle formed by two vertices and the orthocenter of a given triangle is equal to the circumcircle of the given triangle



6.114

A vertex of a triangle is the midpoint of the arc determined on its circumcircle by the two altitudes, produced, issued from the two other vertices

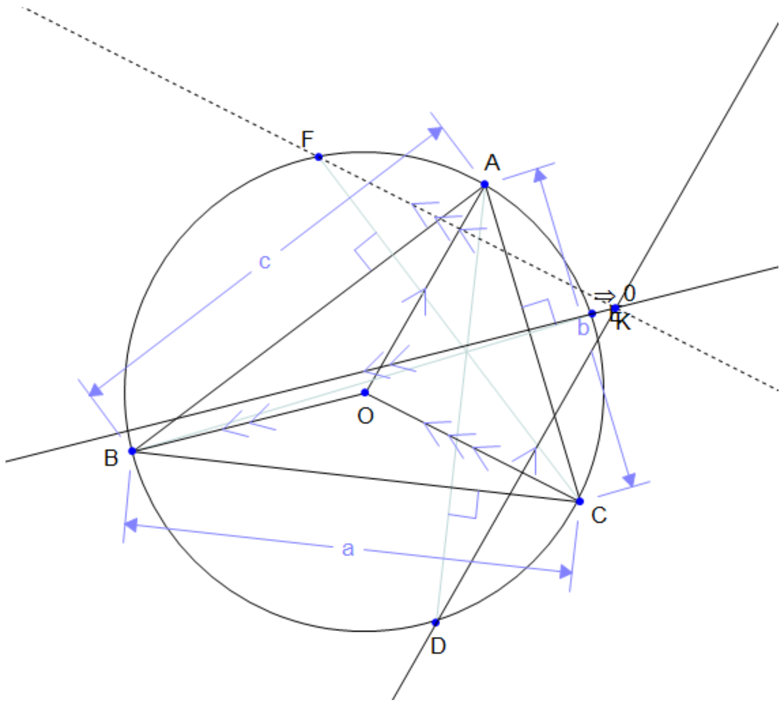


We check that O lies on the perpendicular from C to IK.

The result was not so reasonable when the model was constrained by 3 line lengths.

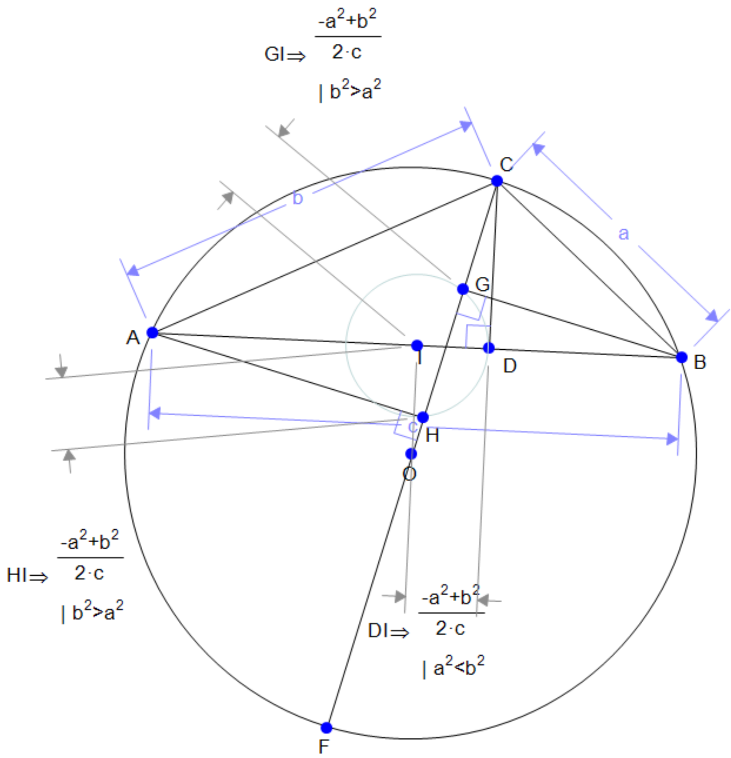
6.115

If O is the circumcenter and H the orthocenter of a triangle ABC, and AH, BH, CH meet the circumcircle in D, E, F, prove that the parallels through D, E, F to OA, OB, OC respectively meet in a point.



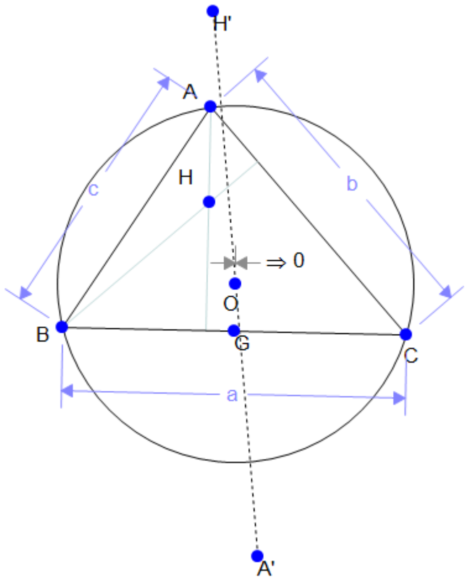
6.116

Show that the foot of the altitude to the base of a triangle and the projections of the ends of the base on the circumdiameter passing through the opposite vertex of the triangle determine a circle having for center the midpoint of the base,



6.117

Show that the symmetric of the orthocenter of a triangle with respect to a vertex, and the symmetric of that vertex with respect to the midpoint of the opposite side, are collinear with the circumcenter of the triangle.

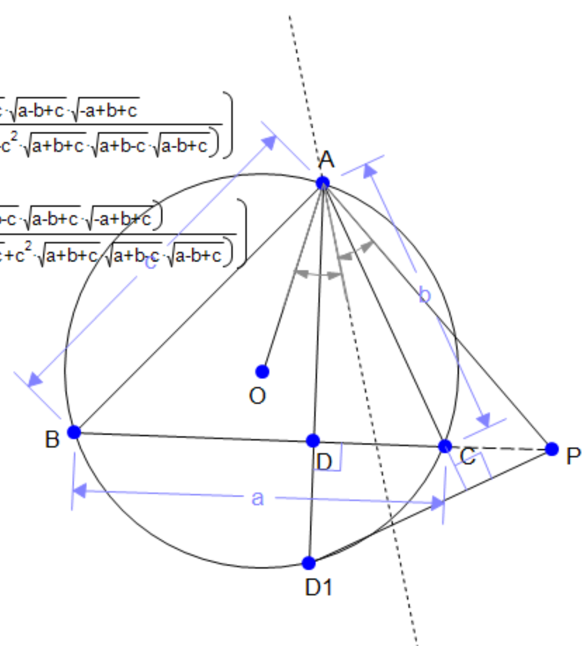


6.118

If D1 is the second point of intersection of the altitude AD of the triangle ABC with the circumcircle, center O, and P is the trace on BC of the perpendicular from D1 to AC. Show that the lines AP, AO make equal angles with the bisector of the angle DAC

$$\Rightarrow \arctan\left(\frac{a^2 \cdot b^2 - b^4 + a^2 \cdot c^2 + 2 \cdot b^2 \cdot c^2 - c^4 - a \cdot b \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{a^3 \cdot b - a \cdot b^3 + a \cdot b \cdot c^2 + \sqrt{-a+b+c} \cdot \left(b^2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} - c^2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c}\right)}\right)$$

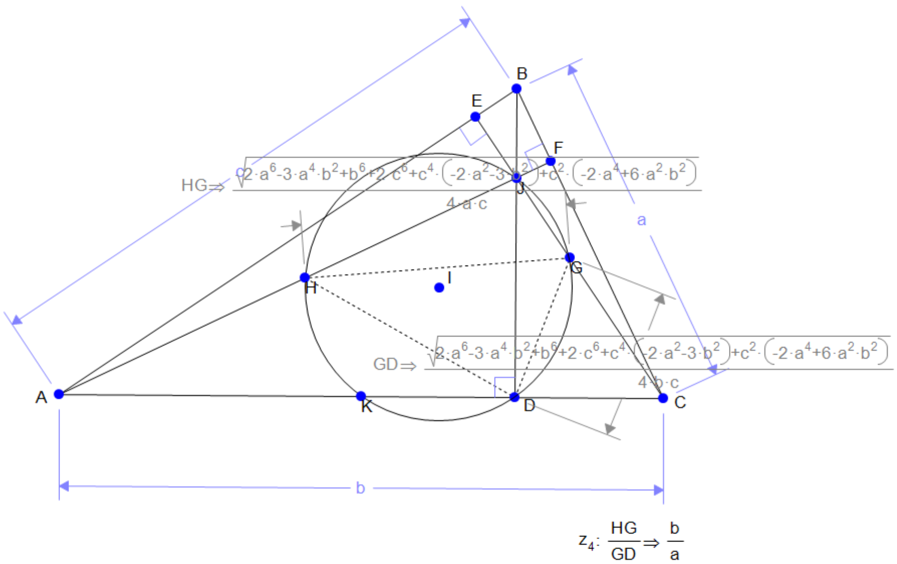
$$z_1 \Rightarrow -\arctan\left(\frac{-\left(-a^2 \cdot b^2 + b^4 - a^2 \cdot c^2 - 2 \cdot b^2 \cdot c^2 + c^4 + a \cdot b \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}\right)}{-a^3 \cdot b + a \cdot b^3 - a \cdot b \cdot c^2 + \sqrt{-a+b+c} \cdot \left(-b^2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} + c^2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c}\right)}\right)$$



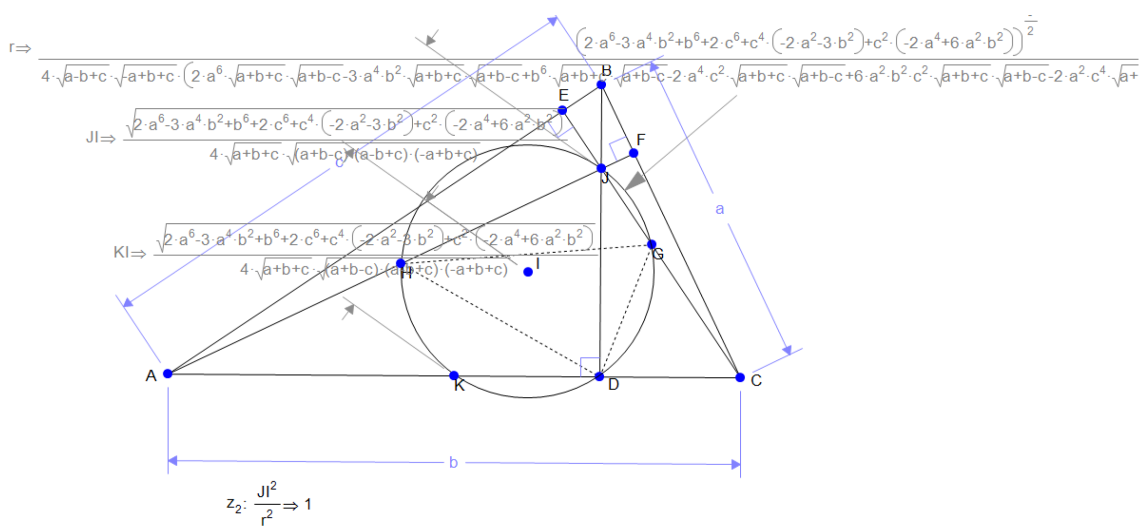
6.119

Show that the triangle formed by the foot of the altitude to the base and the midpoints of the altitudes to the lateral sides is similar to the given triangle , its circumcircle passes through the orthocenter of the given triangle and through the midpoint of its base.

First the similarity

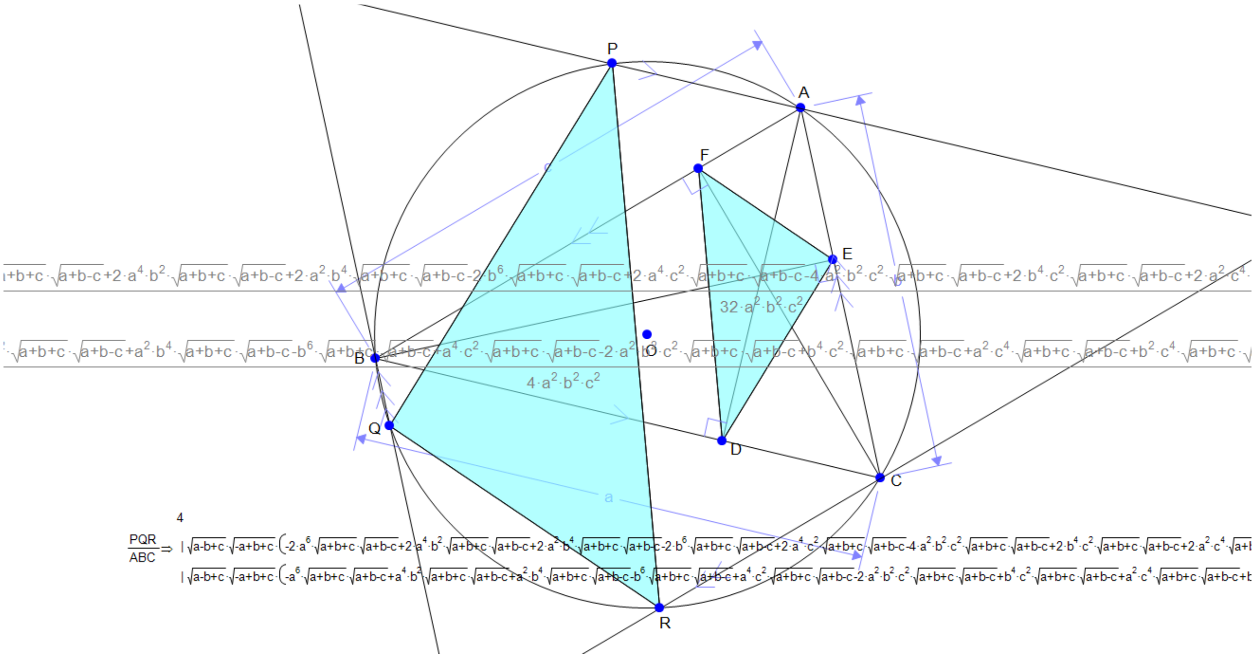


Now the points lying on a circle



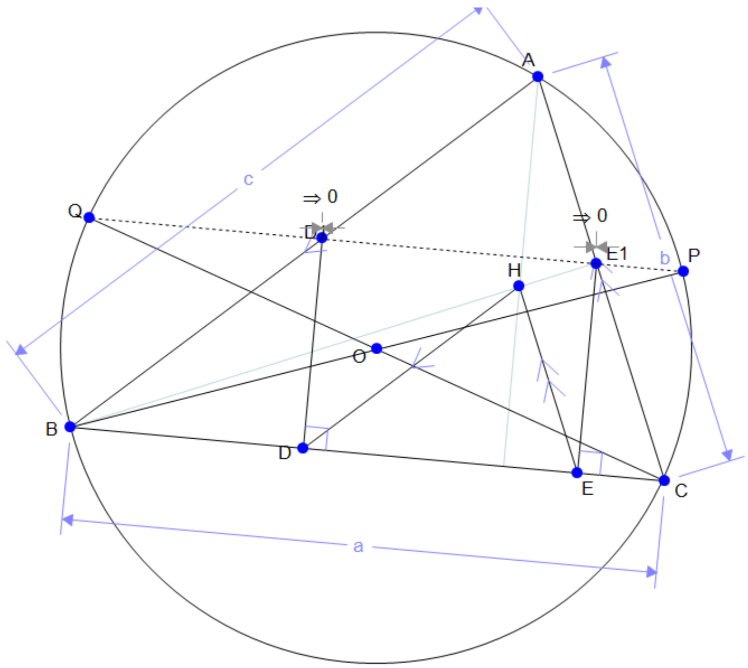
6.120

The sides of the anticomplementary triangle of the triangle ABC meet the circumcircle of ABC in the points P, Q, R. Show that the area of the triangle PQR is equal to four times the area of the orthic triangle of ABC



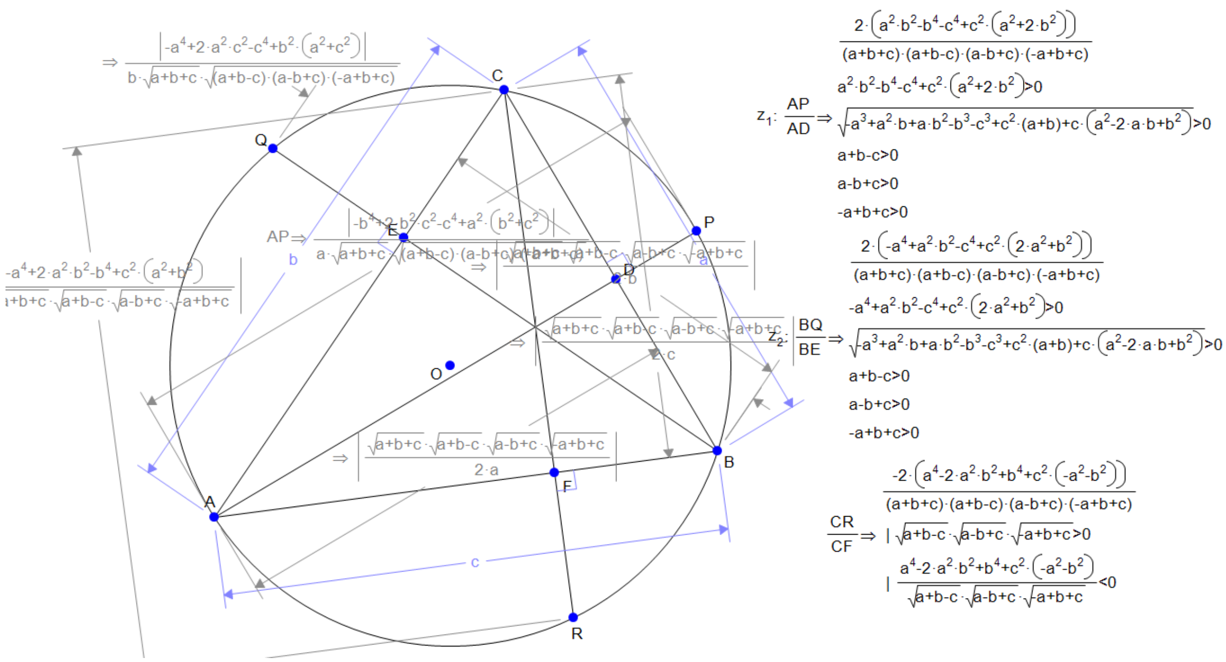
6.121

Through the orthocenter of the triangle ABC parallels are drawn to the sides AB, AC, meeting BC in D, E. The perpendiculars to BC at D, E meet AB, AC in two points D1, E1 which are collinear with the diametric opposites of B, C on the circumference of ABC.



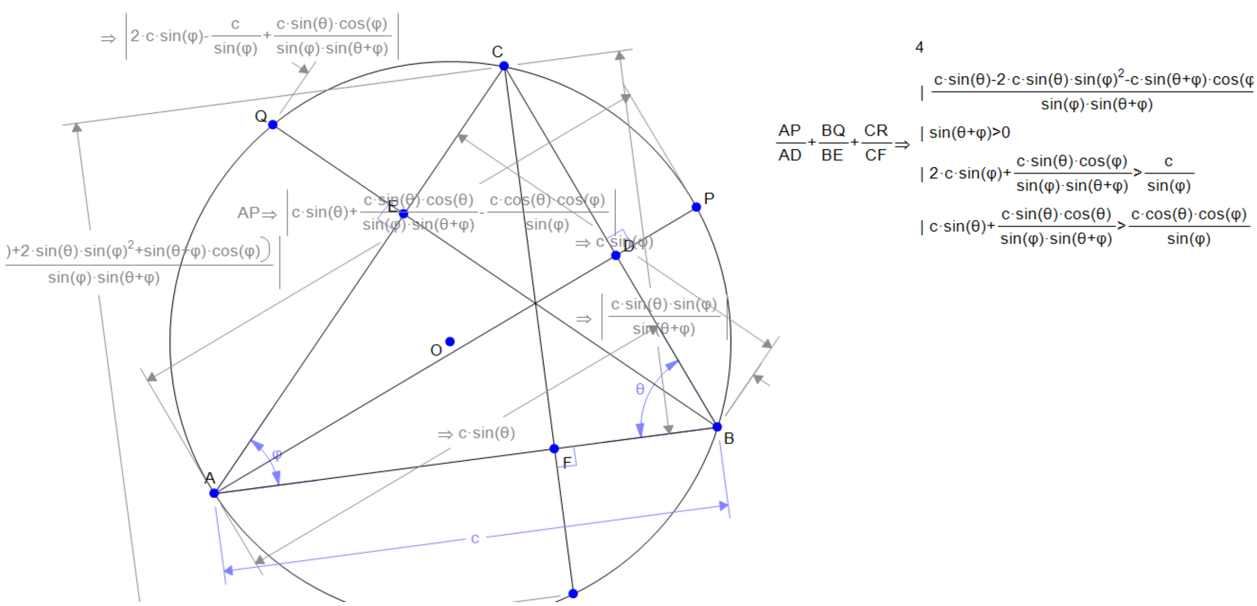
6.122

If the altitudes AD, BE, CF of the triangle ABC meet the circumcircle of ABC in P, Q, R, show that we have $\frac{AP}{AD}+\frac{BQ}{BE}+\frac{CR}{CF}=4$



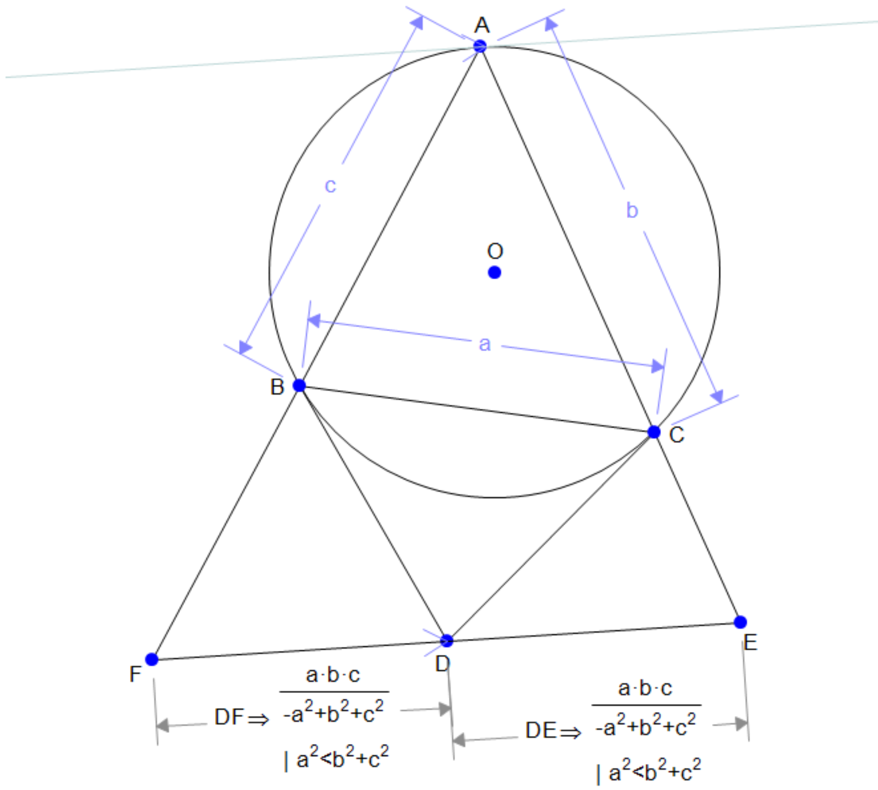
We get as far as the individual ratios, and although one can see that adding these would be on a common denominator, when you create the sum as an expression … it ends up not simplifying (because we go back to the original expression…

But re-constrain using angles and we get the solution we are looking for:



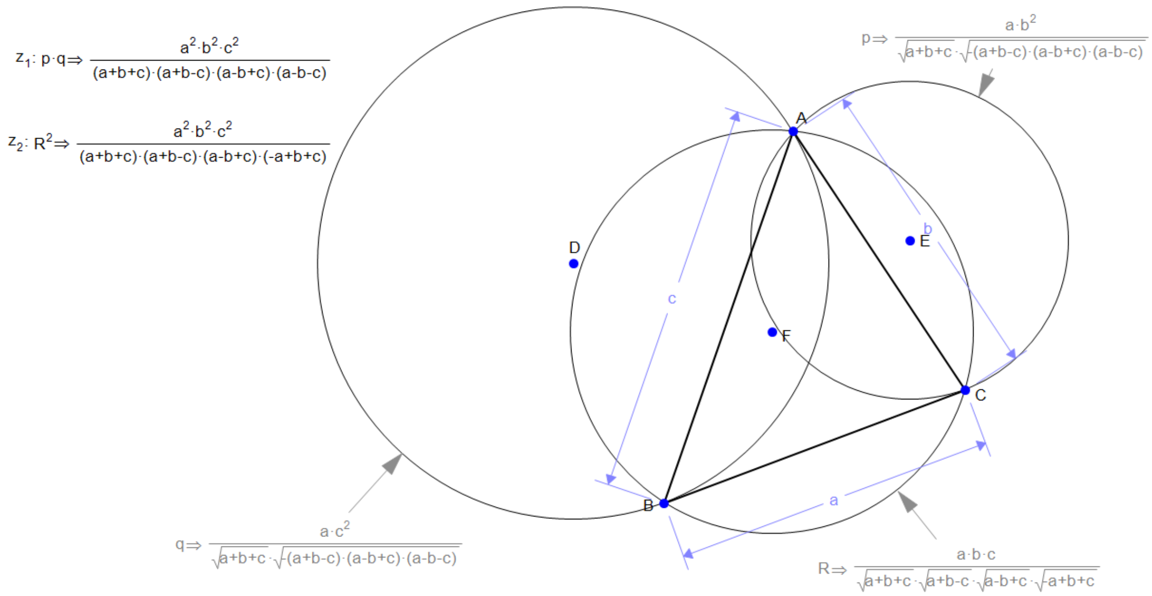
6.123

Through the point of intersection of the tangents DB, DC to the circumcircle (O) of the triangle ABC a parallel is drawn to the line touching (O) at A. If this parallel meets AB, AC in E, F show that D bisects EF.



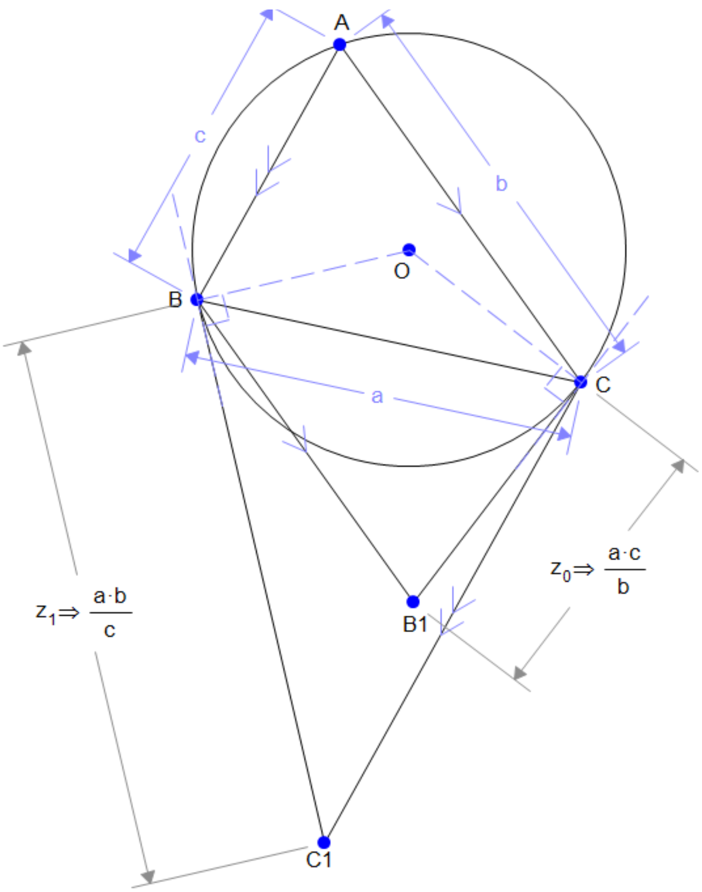
6.124

In a triangle ABC let p and q be the radii of two circles through A touching side BC at B and C respectively. Then p.q=R^2 (where R is the circumradius).



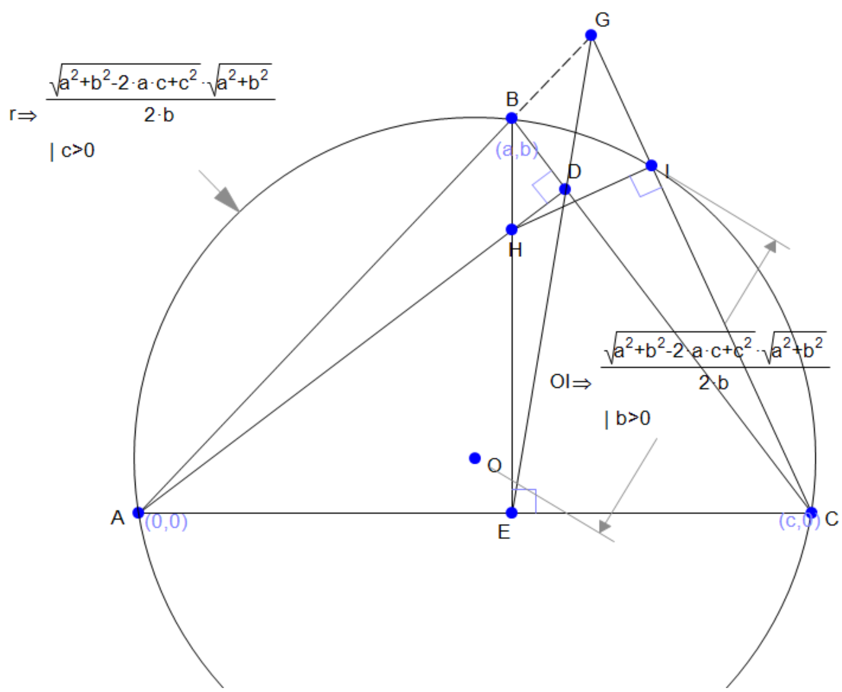
6.125

The parallel to the side AC through the vertex B of the triangle ABC meets the tangent to the circumcircle (O) of ABC at C in B1, and the parallel through C to AB meets the tangent to (O) at B in C1. Prove that BC^2=BC1.B1C



6.126

Show that the foot of the perpendicular from the orthocenter of a triangle upon the line joining a vertex to the point of intersection of the opposite side with the corresponding side of the orthic triangle lies on the circumcircle of the triangle.



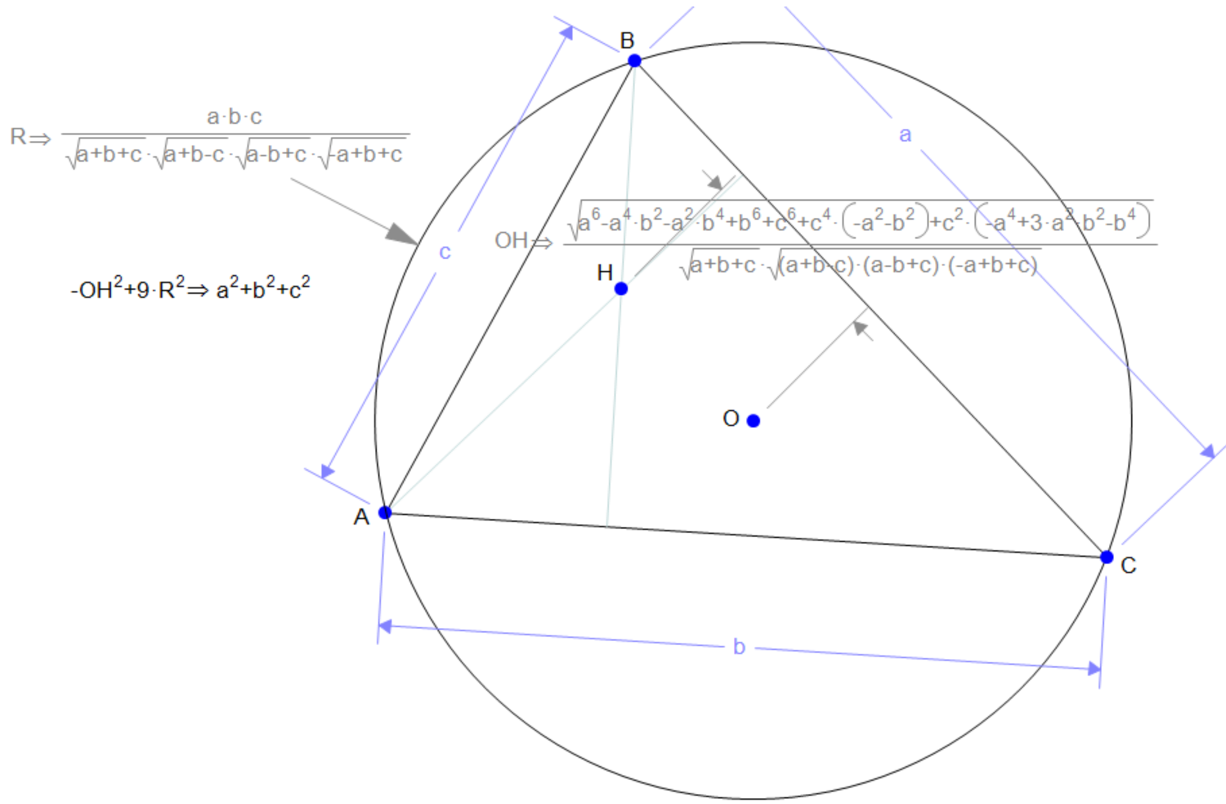
This one was nasty when I constrained with line lengths. Coordinates did the trick, though.

2.3.4 The Euler Line

The circumcenter O, orthocenter H and centroid G of a given triangle are collinear and the line is called the Euler Line of the triangle.

6.127

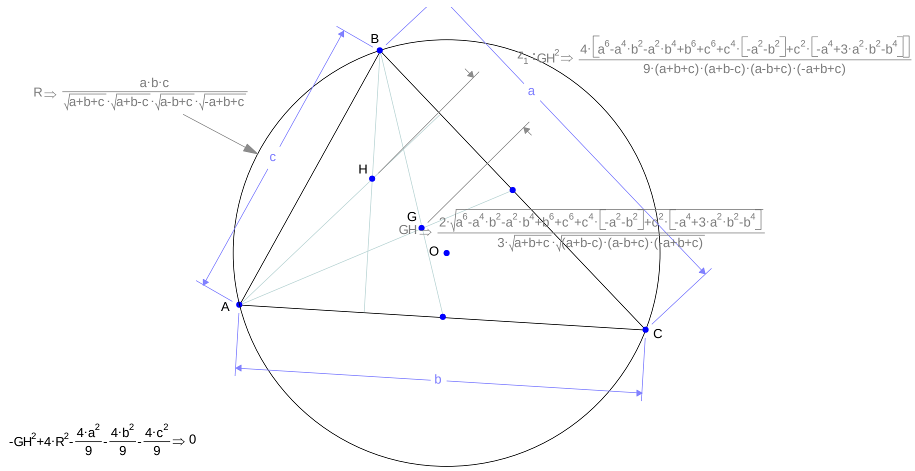
Let O and H be the circumcenter and orthocenter of a triangle ABC. Show that $OH^2=9R^2-a^2-b^2-c^2$



6.128

With the usual notations for the triangle ABC we have:

$4AO^2=4AB^2+4AC^2+4BC^2+GH^2$

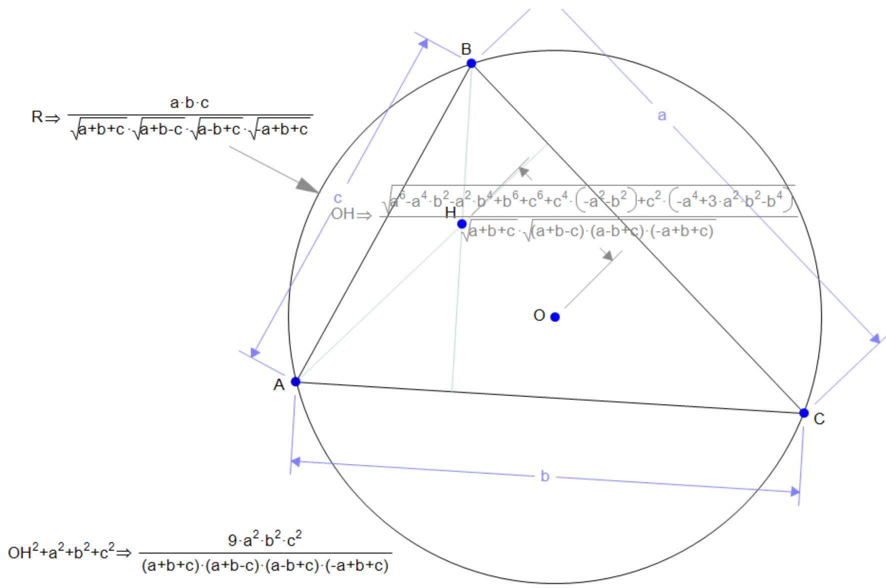


It looks like the problem as stated is wrong and is missing the denominator in the coefficients of AB, AC and BC.

6.129

With the usual notations for the triangle ABC we have:

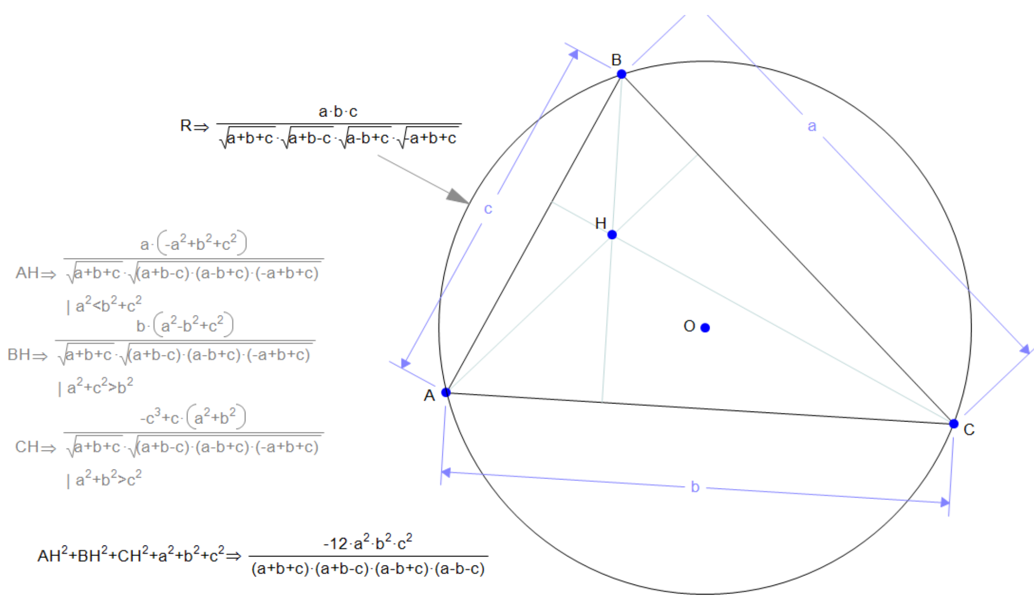
$9AO^2=AB^2+AC^2+BC^2+OH^2$



6.130

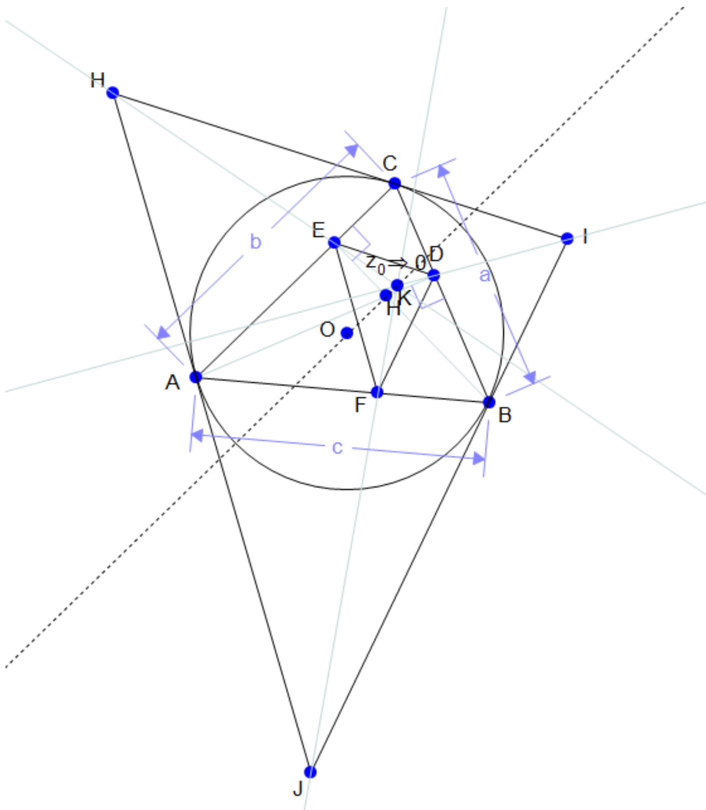
With the usual notations for the triangle ABC we have:

$12AO^2= AB^2+AC^2+BC^2+AH^2+BH^2+CH^2$



6.131

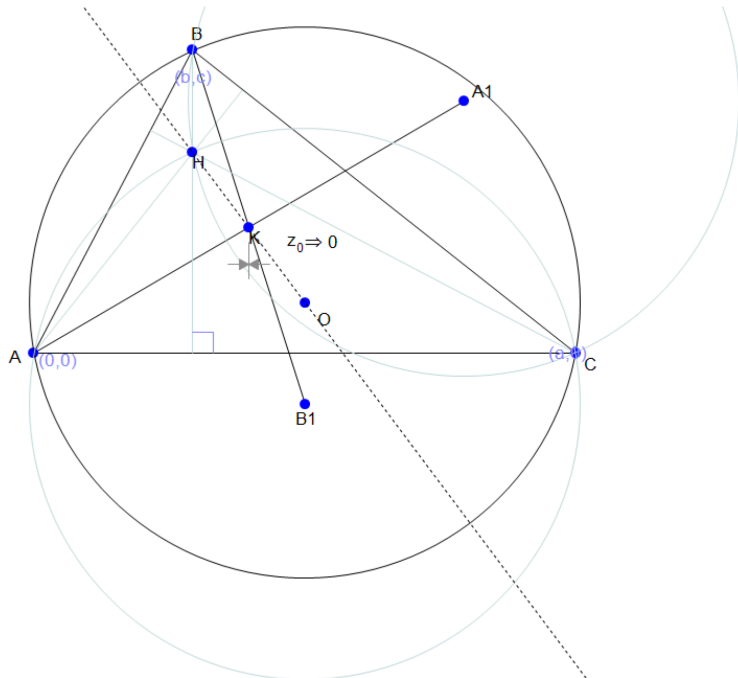
The homothetic center of the orthic and the tangential triangles of a given triangle lies on the Euler line of the given triangle (see 6.106)



We show that the distance of K from the line joining O and H is zero.

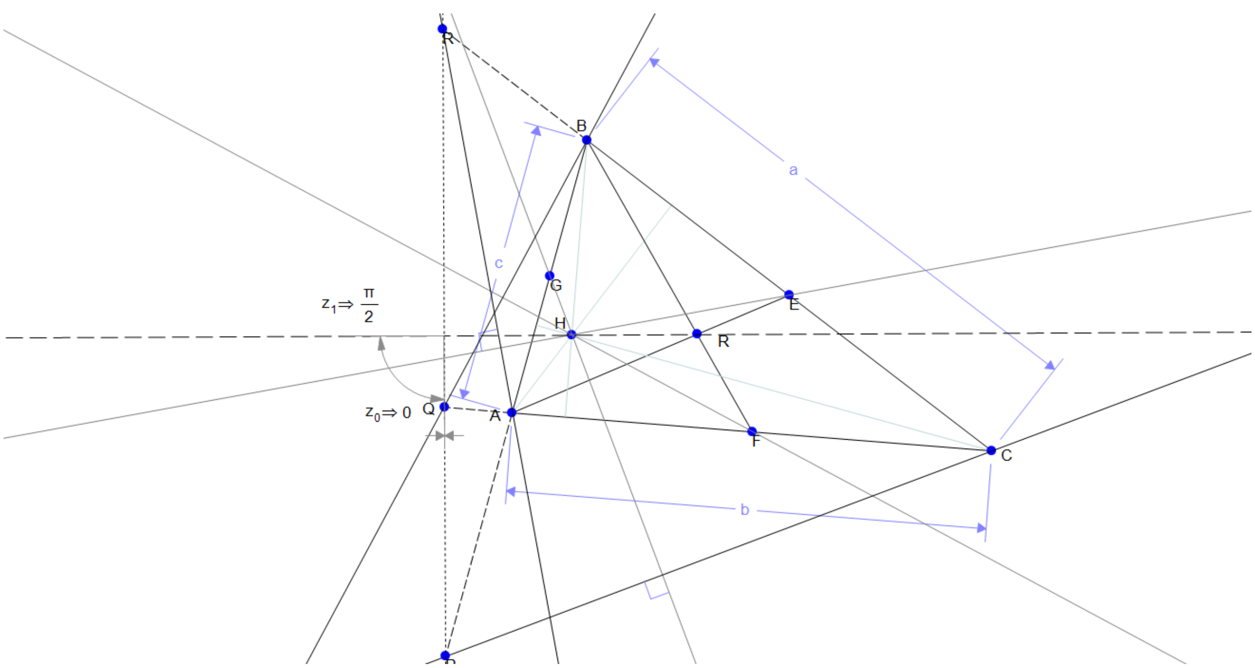
6.132

The Euler Lines of the four triangles of an orthocentric group are concurrent.



6.133

Show that the perpendiculars from the vertices of a triangle to the lines joining the midpoints of the respectively opposite sides to the orthocenter of the triangle meet these sides in three points of a straight line perpendicular to the Euler line of the triangle.

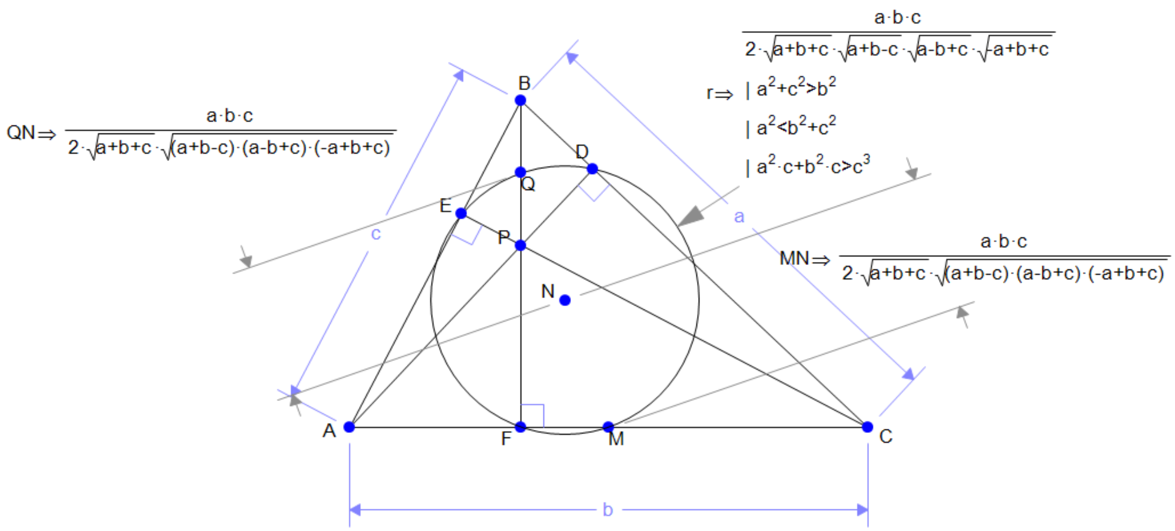


2.3.5 The Nine Point Circle

The midpoints of the segments joining the orthocenter of a triangle to its vertices are called the Euler Points of the triangle. The three Euler Points determine the Euler Triangle.

6.134 The Nine Point Circle Theorem

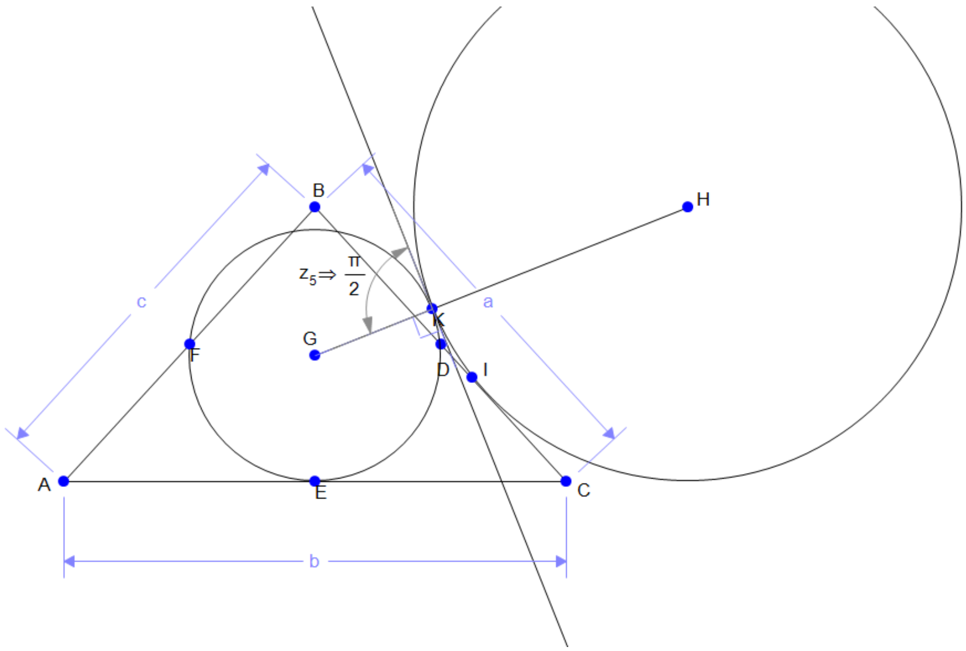
In a triangle, the midpoints of the sides, the feet of the altitudes and the Euler Points lie on the same circle.



The circle (N) id through the feet of the altitudes. Q is an Euler point and M a midpoint. We show that MN and QN are the same as the radius.

6.135 Feuerbach's Theorem

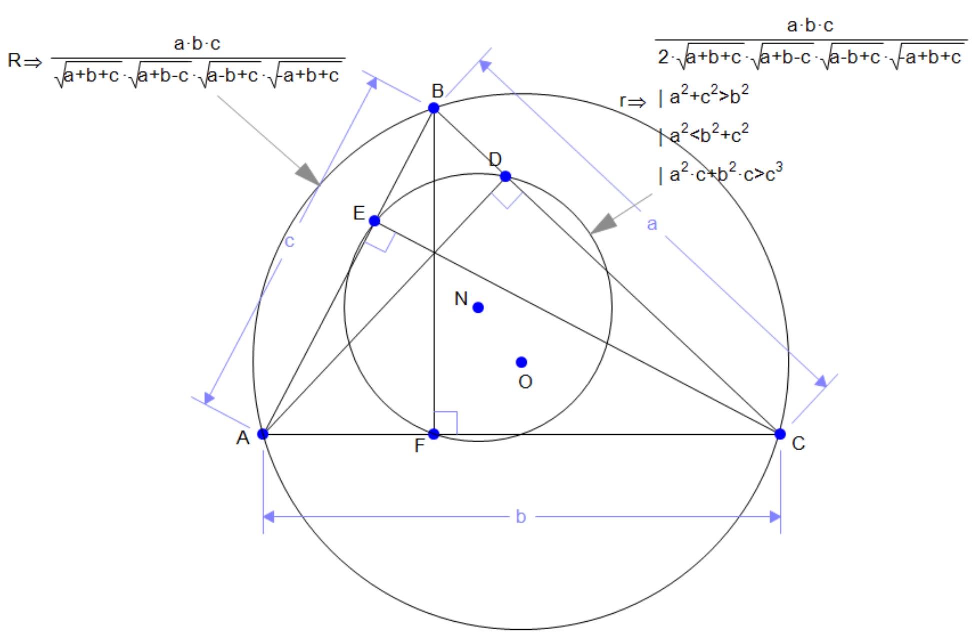
The nine-point circle of a triangle touches each of the four tritangent circles of the triangle.



K is the intersection of the lines joining the centers of the circles with the nine point circle. We show that this is perpendicular to the tangent at K.

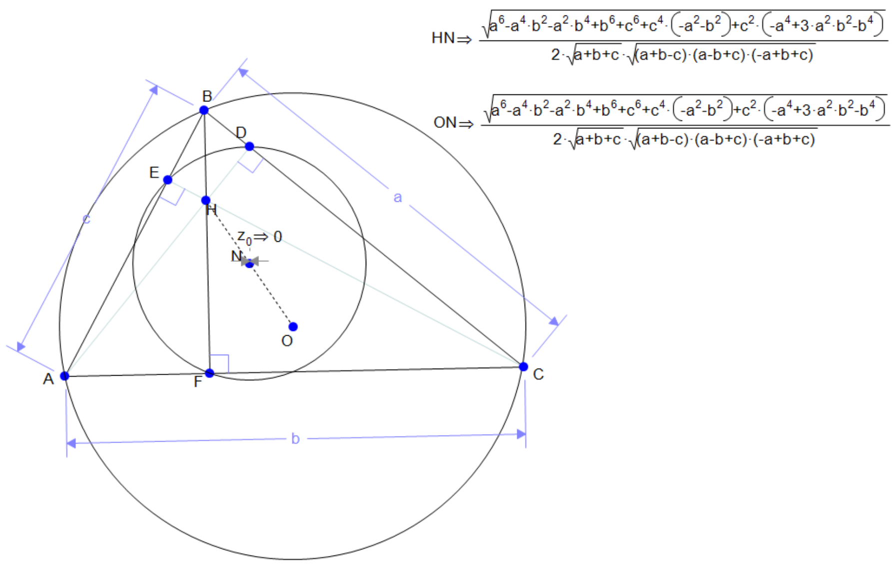
6.136

The radius of the nine-point circle is equal to half the circumradius of the triangle



6.137

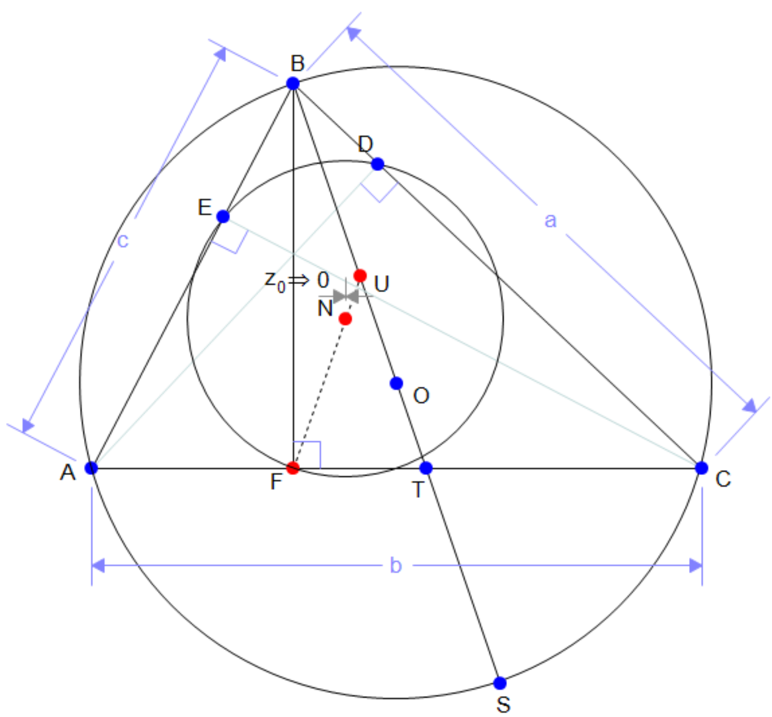
The nine-point circle center lies on the Euler line midway between the circumcenter and the orthocenter.



We show that N lies on HO and that HN=NO.

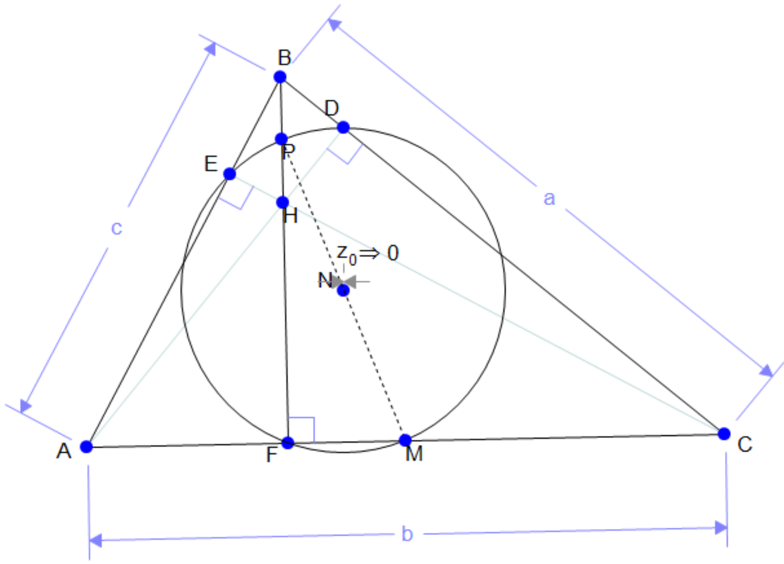
6.138

Show that the foot of the altitude of a triangle on a side, the midpoint of the segment of the circumdiameter between this side and the opposite vertex and the nine point center are collinear



6.139

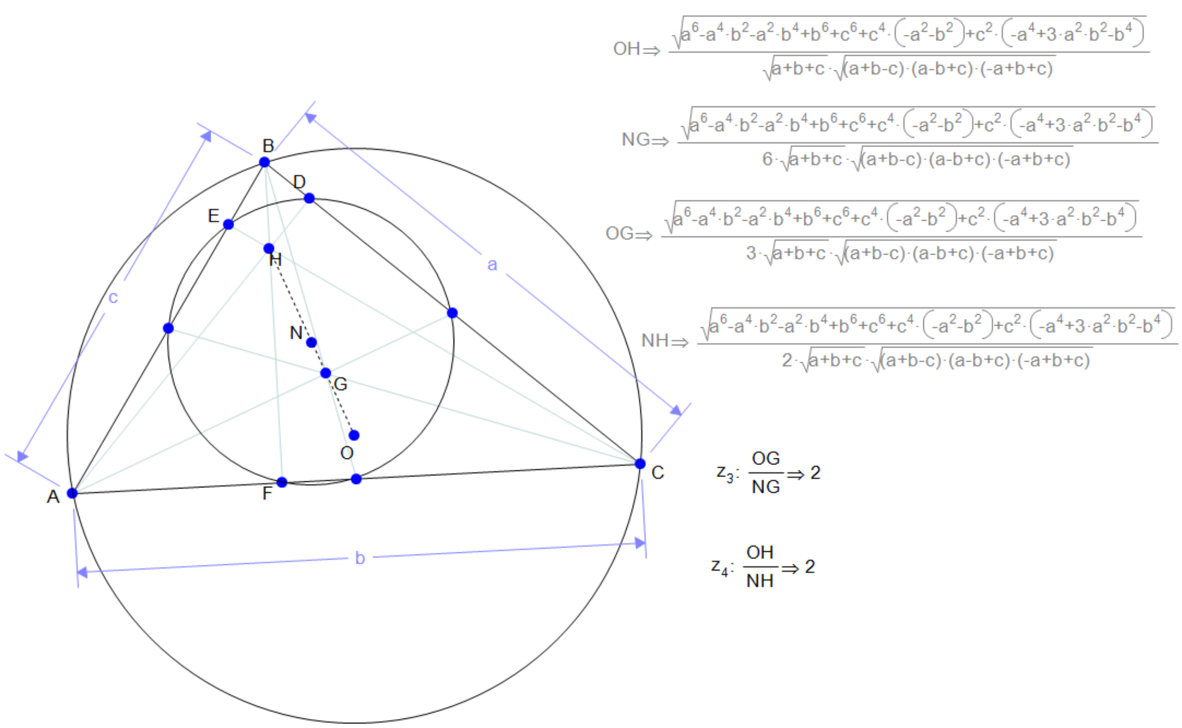
The center of the nine-point circle is the midpoint of a Euler point and the midpoint of the opposite side.



Both points lie on the 9-point circle, so all we need to show is that the center lies on the chord between the points.

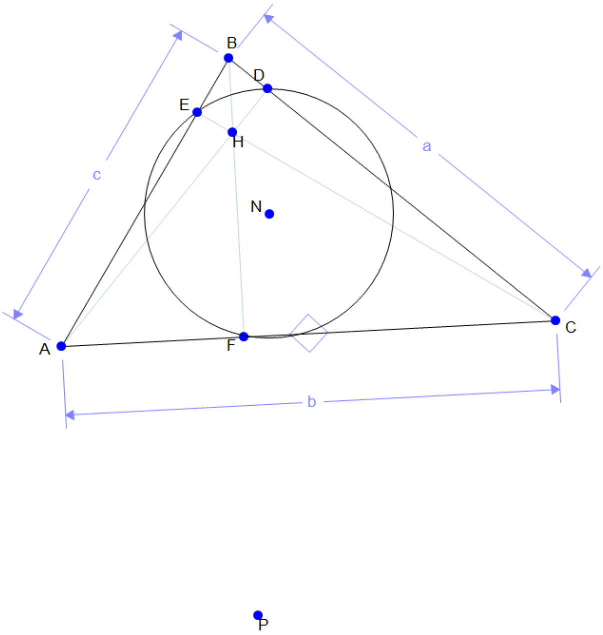
6.140

The two pairs of points O and N, G and H separate themselves harmonically.



6.141

If P is the symmetric of the vertex A with respect to the opposite side BC, show that HP is equal to 4 times the distance of the nine-point center from BC



$$HP \Rightarrow \frac{-a^4+a^2\cdot b^2-c^4+c^2\cdot\left(2\cdot a^2+b^2\right)}{b\cdot\sqrt{a+b+c}\cdot\sqrt{(a+b-c)\cdot(a-b+c)\cdot(-a+b+c)}}$$

$$-c^5+c^3\cdot\left(2\cdot a^2+b^2\right)+c\cdot\left(-a^4+a^2\cdot b^2\right)>0$$

$$NAC \Rightarrow \frac{-\left(a^4-a^2\cdot b^2+c^4+c^2\cdot\left(-2\cdot a^2-b^2\right)\right)}{4\cdot b\cdot\sqrt{a+b+c}\cdot\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}$$

$$\frac{a^4-a^2\cdot b^2+c^4+c^2\cdot\left(-2\cdot a^2-b^2\right)}{\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}<0$$

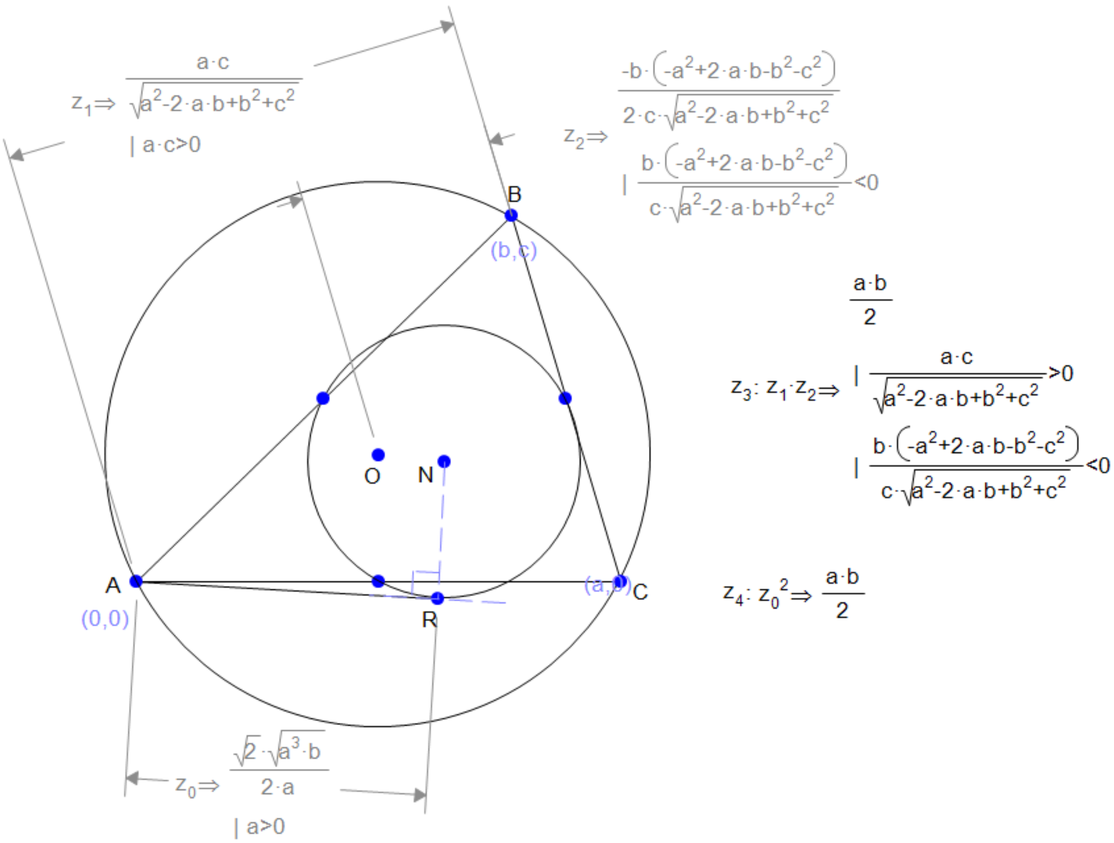
4

$$z_0:\ \frac{HP}{NAC} \Rightarrow \frac{a^4-a^2\cdot b^2-2\cdot a^2\cdot c^2-b^2\cdot c^2+c^4}{\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}<0$$

$$-a^4\cdot c+2\cdot a^2\cdot c^3-c^5+b^2\cdot\left(a^2\cdot c+c^3\right)>0$$

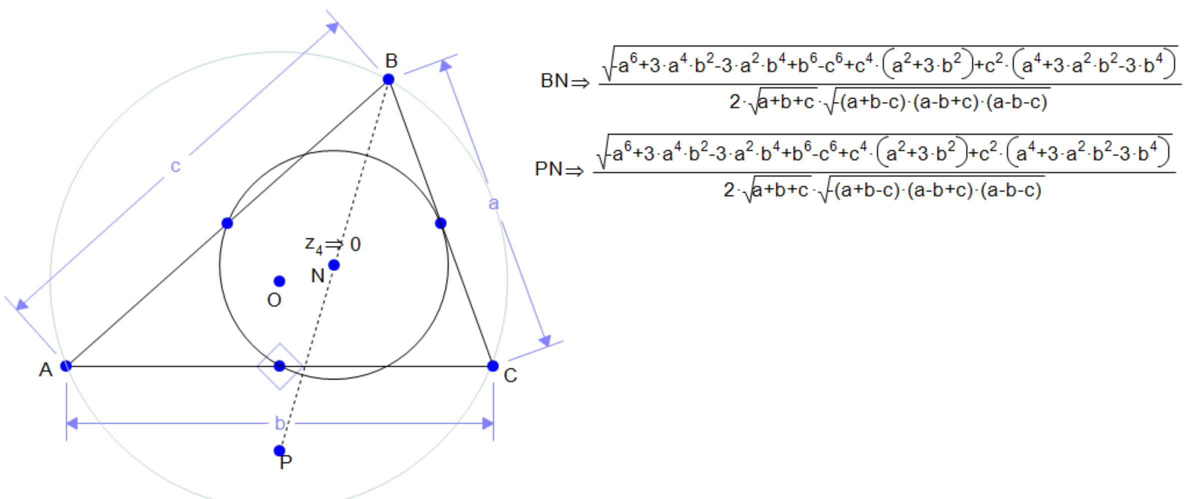
6.142

Show that the square of the tangent from a vertex of a triangle to the nine-point circle is equal to the altitude issued from that vertex multiplied by the distance of the opposite side from the circumcenter.



6.143

Show that the symmetric of the circumcenter of a triangle with respect to a side coincides with the symmetric of the vertex opposite the side considered with respect to the nine-point center of the triangle.

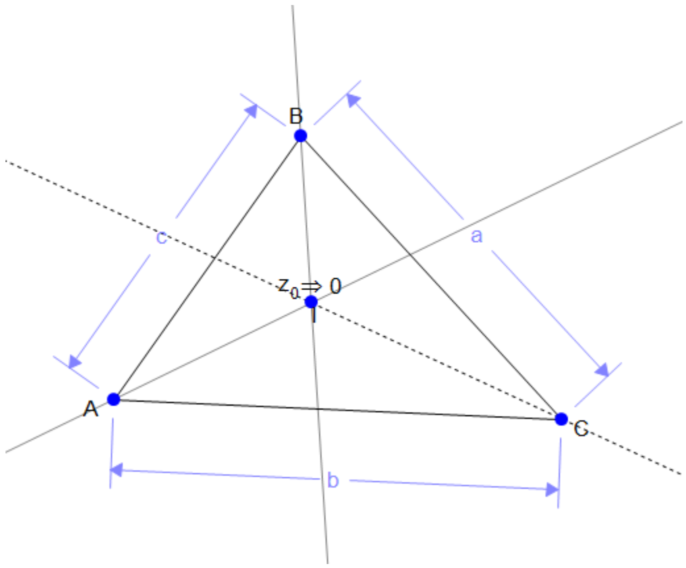


P is the image of O under reflection in AC. We show that N lies on BP and that BN=PN.

2.3.6 Incircles and Excircles

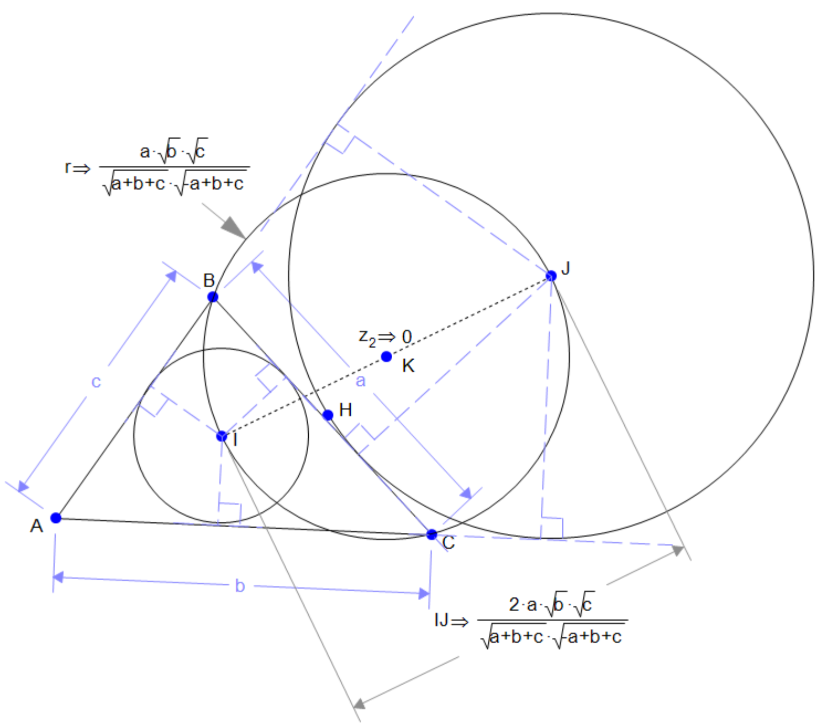
6.144 Theorem of Incenter

The three internal bisectors of the angles of a triangle meet in a point, the incenter I of the triangle



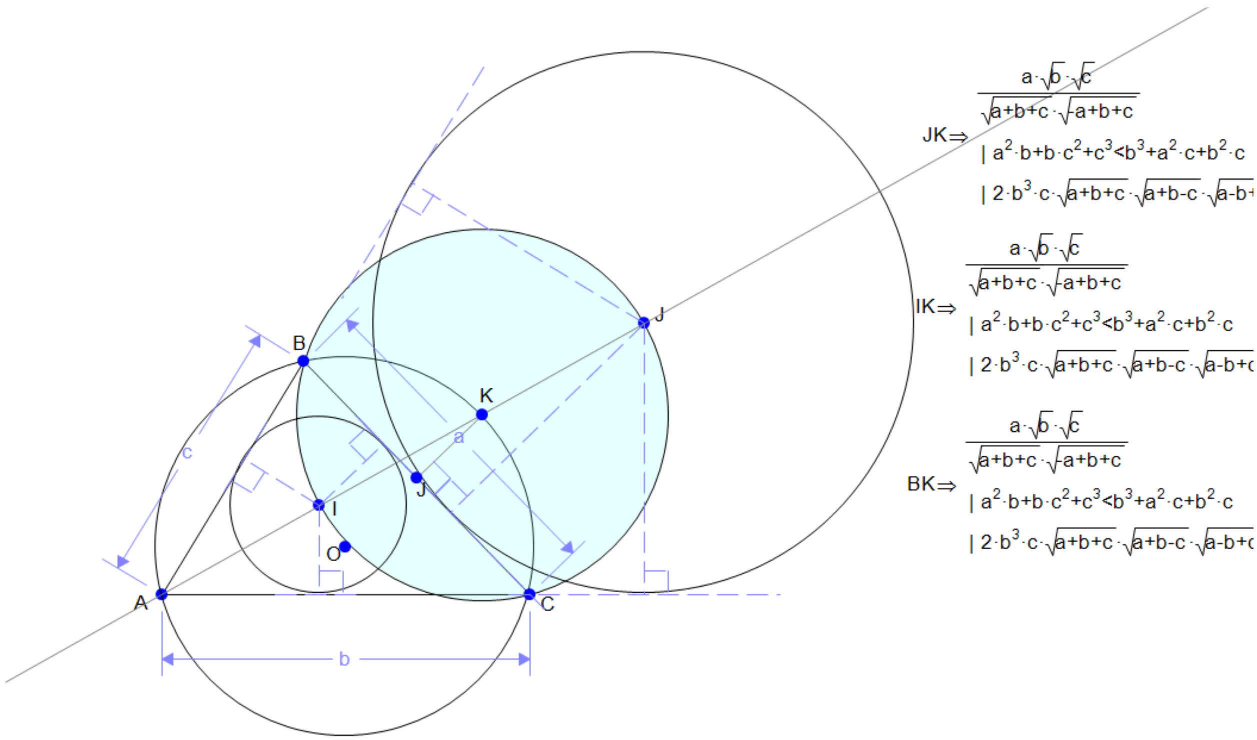
6.145

Two tritangent centers of a triangle are the ends of a diameter of a circle passing through the two vertices of the triangle which are not collinear with the centers considered.



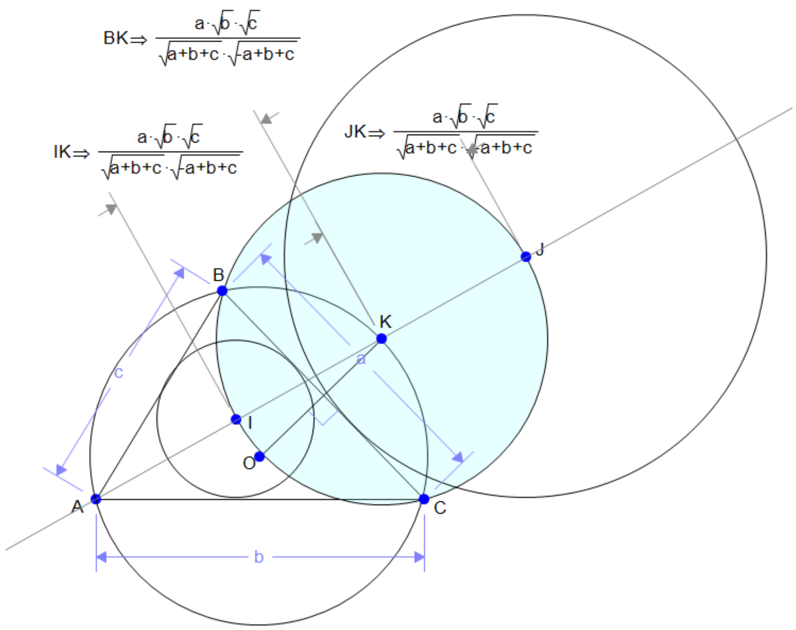
6.146

The four tritangent centers of a triangle lie on six circles which pass through the pairs of vertices of the triangle and have for their centers the midpoints of the arcs subtented by the respective sides of the respective sides of the tringle on its circumcircle.



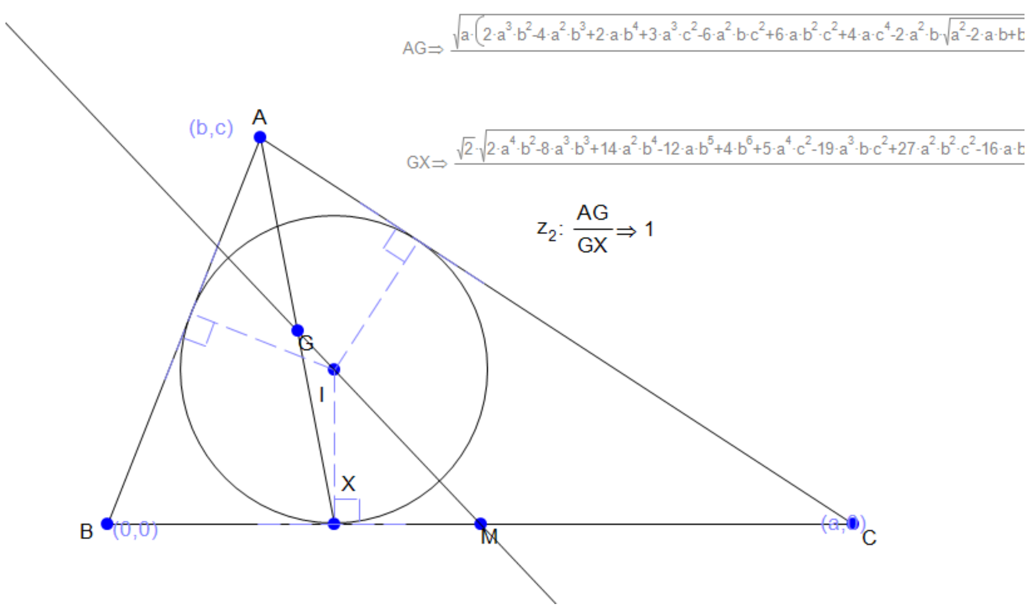
We use the same way of creating the center K as the book, relying on a theorem (6.111) that the angle bisector hits the circumcircle in the point described.

However, we can use this formulation (putting K on the circumcircle and constraining OK to be perpendicular to BC:



6.147

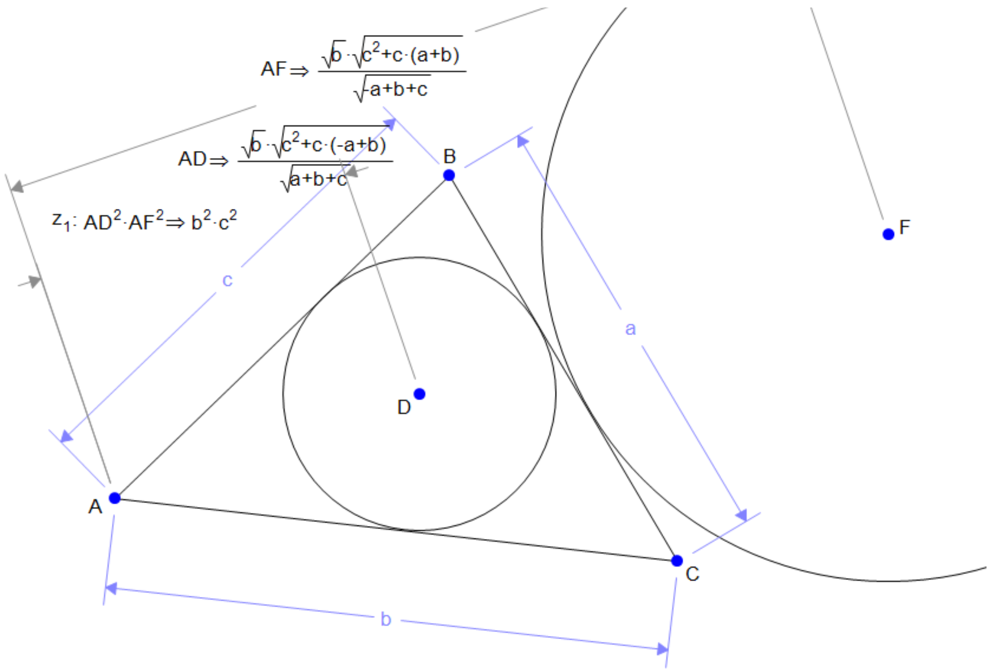
Let incircle (with center I) of triangle ABC touch the side BC at X and M be the midpoint of this side. Then line MI bisects AX.



This one required the model to be pushed so that BC were on the x axis.

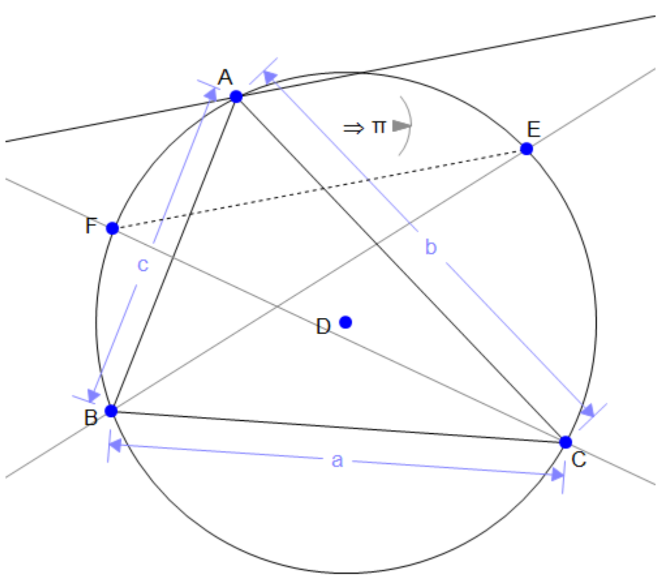
6.148

The product of the distances of two tritangent centers of a triangle from the vertex of the triangle collinear with them is equal to the product of the two sides of the triangle passing through the vertex considered.



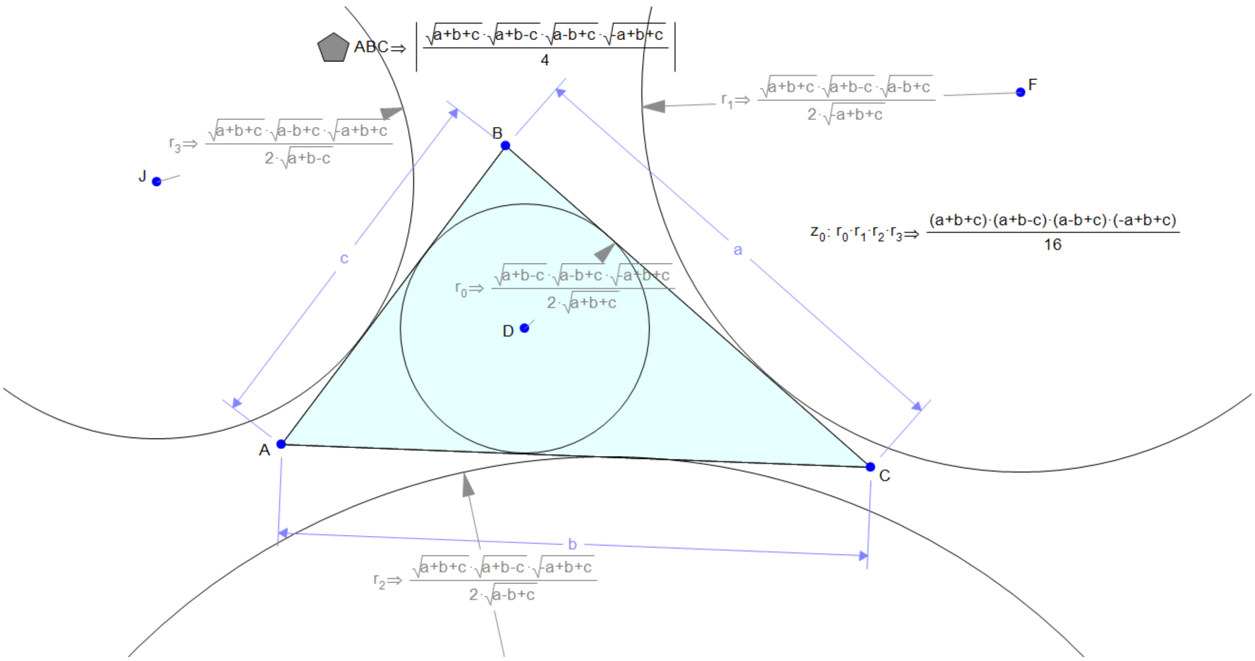
6.149

Show that the external bisector of an angle of a triangle is parallel to the line joining the points where the circumcircle is met by the external (internal) bisectors of the other two angles of the triangle



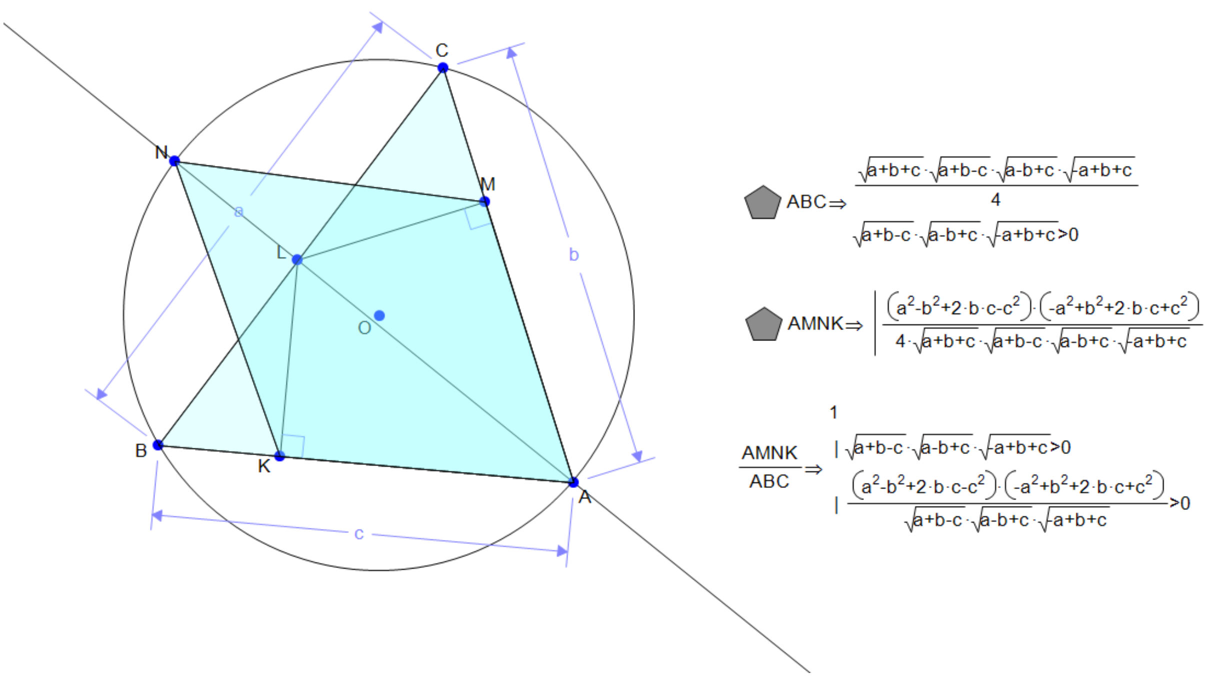
6.150

The product of the four tritangent radii of a circle is equal to the square of its area



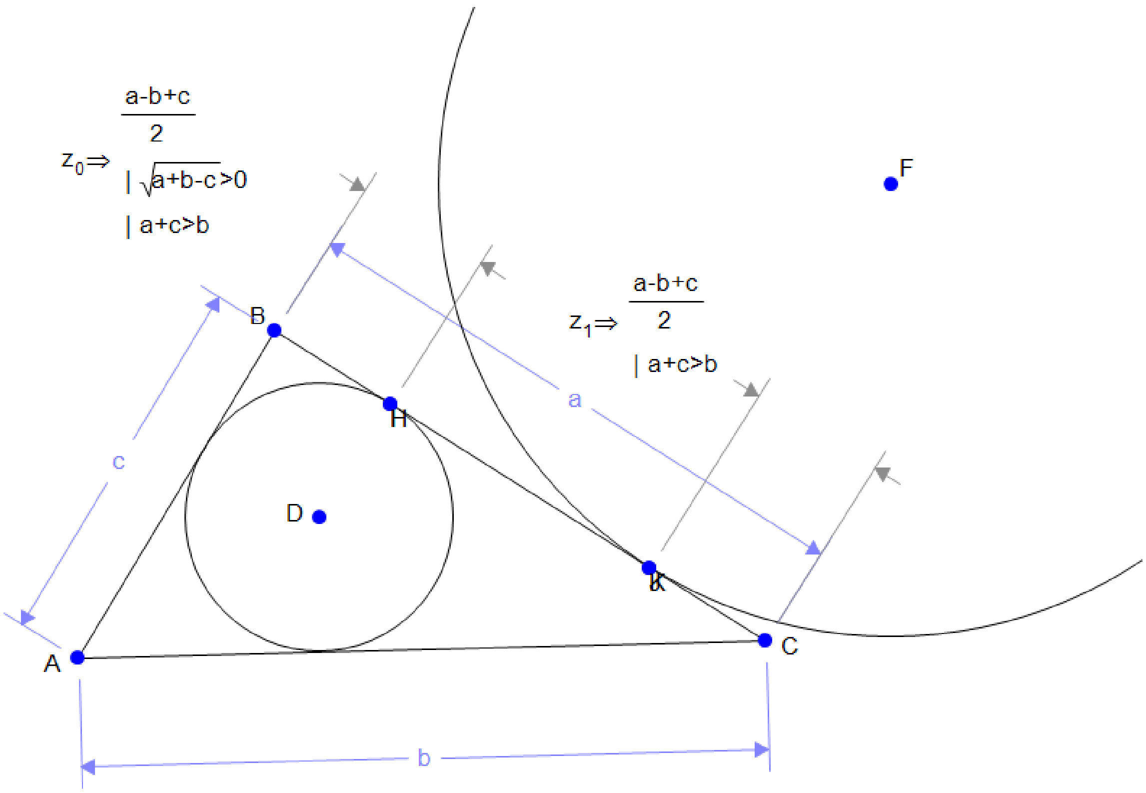
6.151

In a triangle ABC, the bisector of angle A meets BC at L and the circumcircle of triangle ABC at N. Thhhe feet of the perpendiculars from L to AB and AC are K and M. Show that the area of ABC equals the area of AKNM.



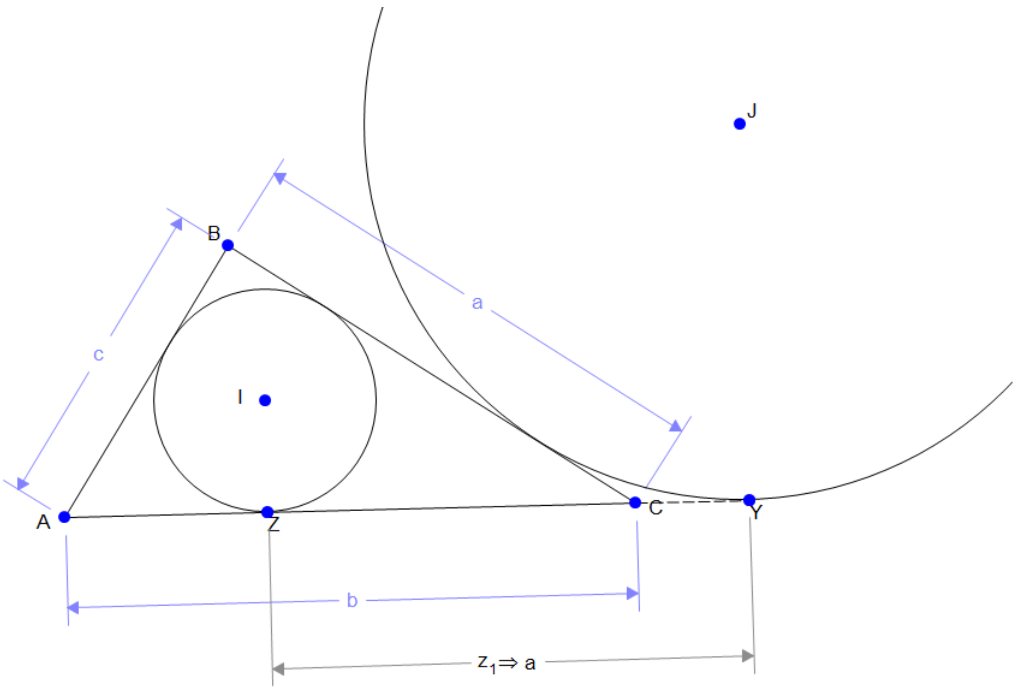
6.152

The points of contact of a side of a triangle with the incircle and the excircle relative to this side are two isotomic points



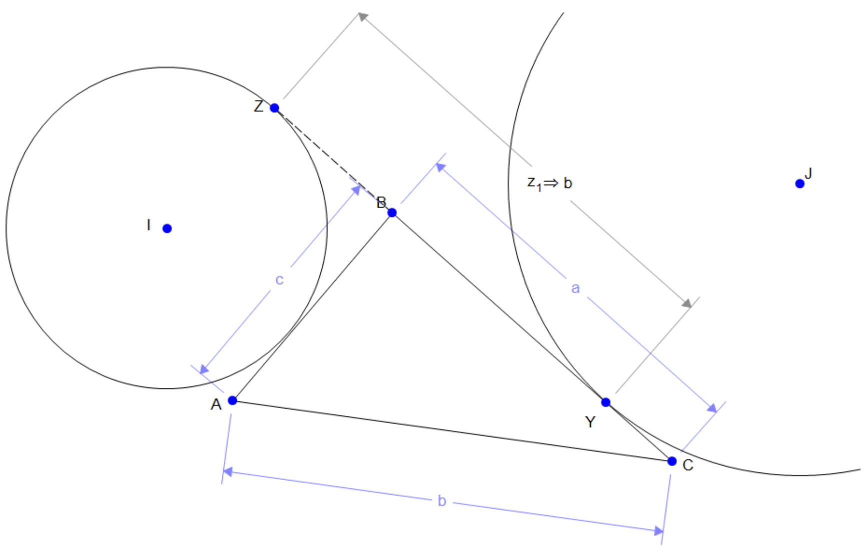
6.153

(I) is the incircle, (J) the excircle defined by side BC. The distance between the points of contact of (I) and (J) with AC (extended) is the same as the length of BC.



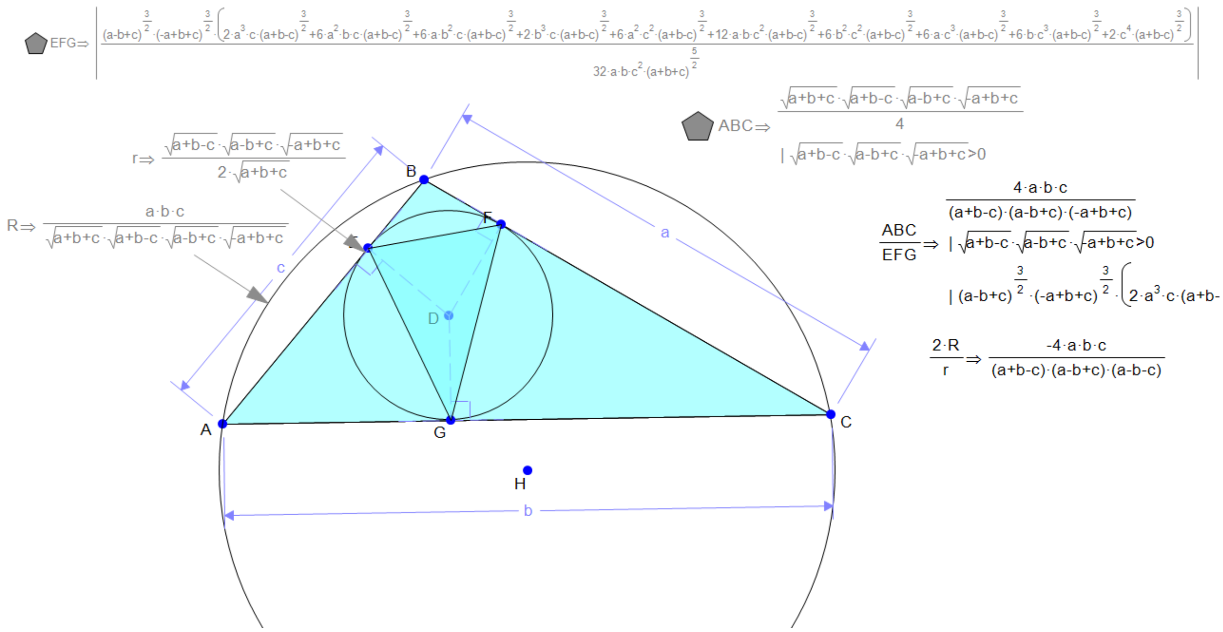
6.154

(I) is the excircle defined by AB, (J) the excircle defined by side BC. The distance between the points of contact of (I) and (J) with BC (extended) is the same as the length of AC.



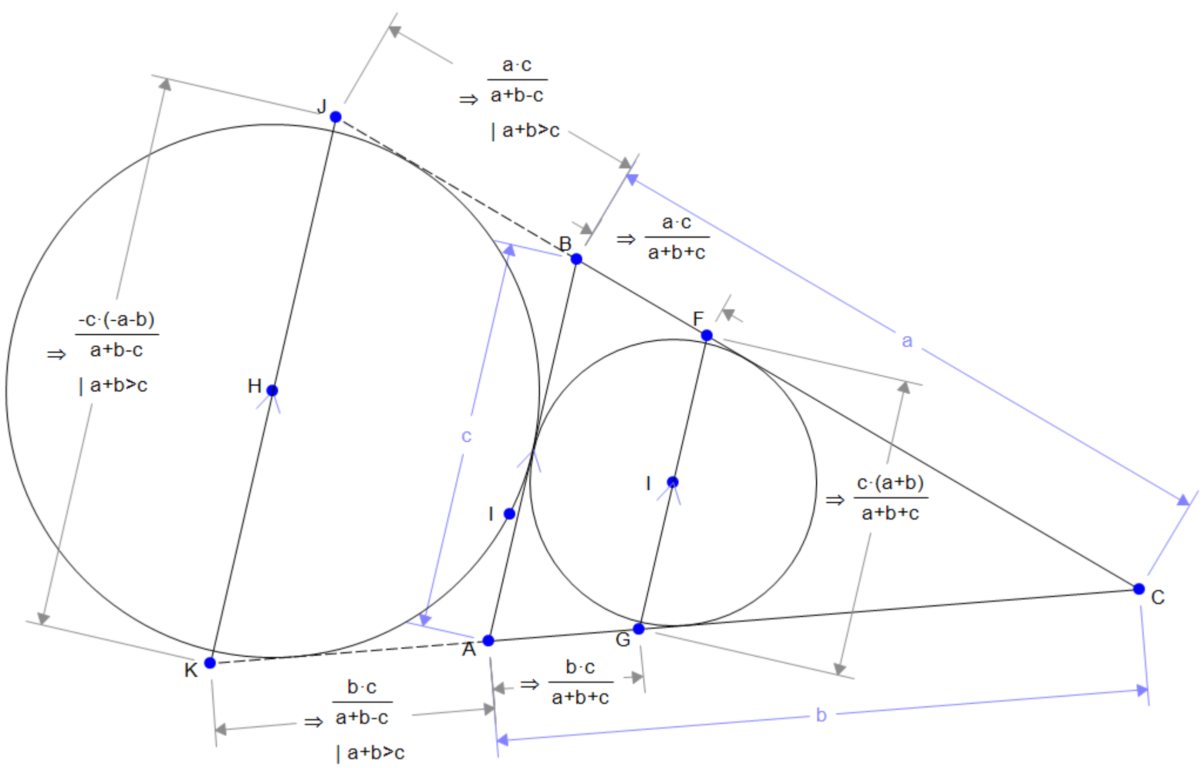
6.155

The ratio of the area of a triangle to the area of the triangle determined by the points of contact of the sides with the incircle is equal to the ratio of the circumdiameter of the given triangle with its inradius.



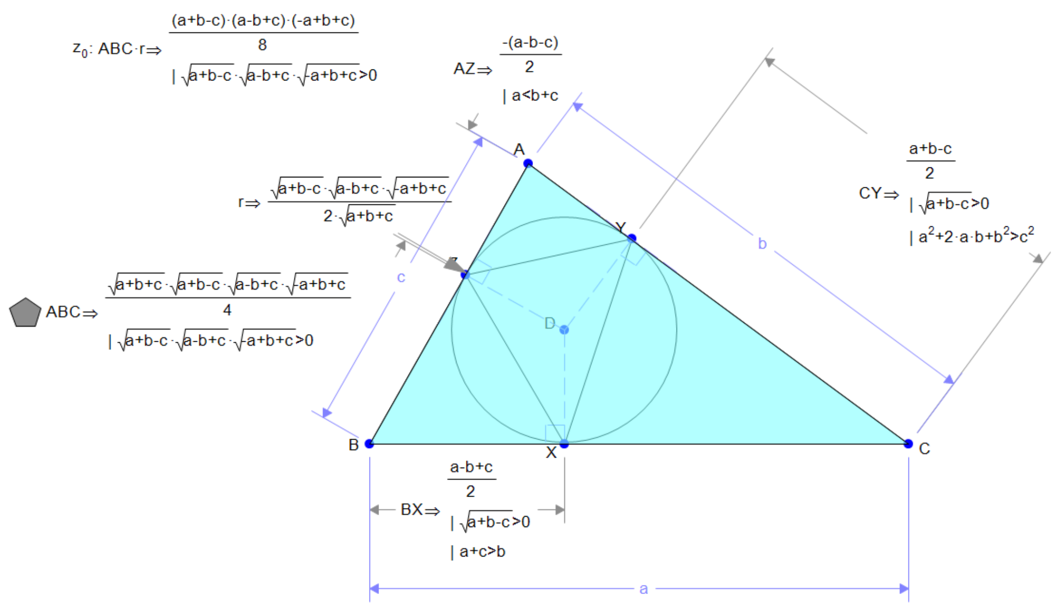
6.156

Show that a parallel through a tritangent center to a side of a triangle is equal to the sum or difference of the other two sides of the triangle between the two parallel lines considered.



6.157

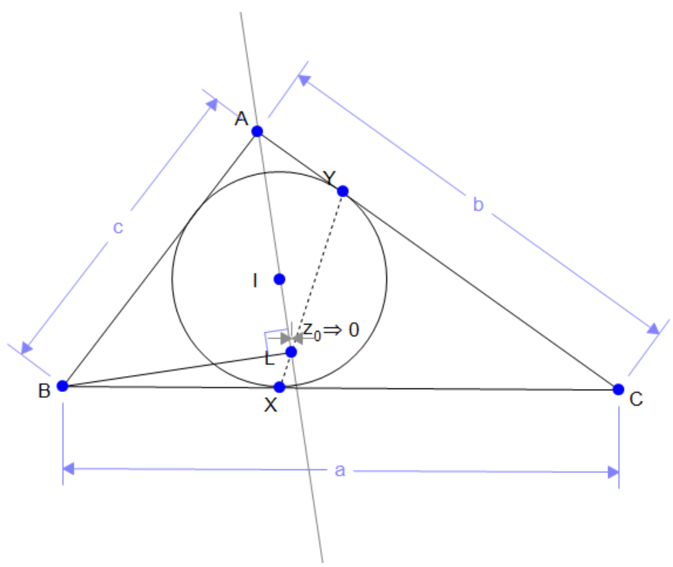
If X, Y and Z are the points of contact between the incircle and the triangle opposite A, B, C respectively. Show that AZ.BX.CY = r times the area of the triangle



Visually we can see that the identity holds, but when we ask Geometry Expressions to compute the product of the lengths, because it goes back to the unsimplified expression, it does not give the good answer!

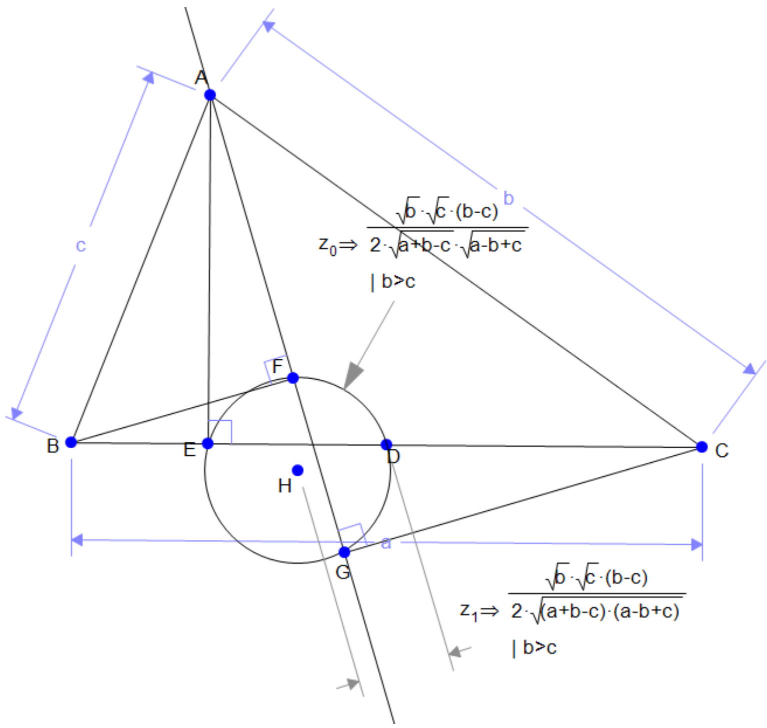
6.158

The projection of the vertex B of the triangle ABC upon the internal bisector of the angle A lies on the line joining the points of contact of the incircle with the sides BC and AC



6.159

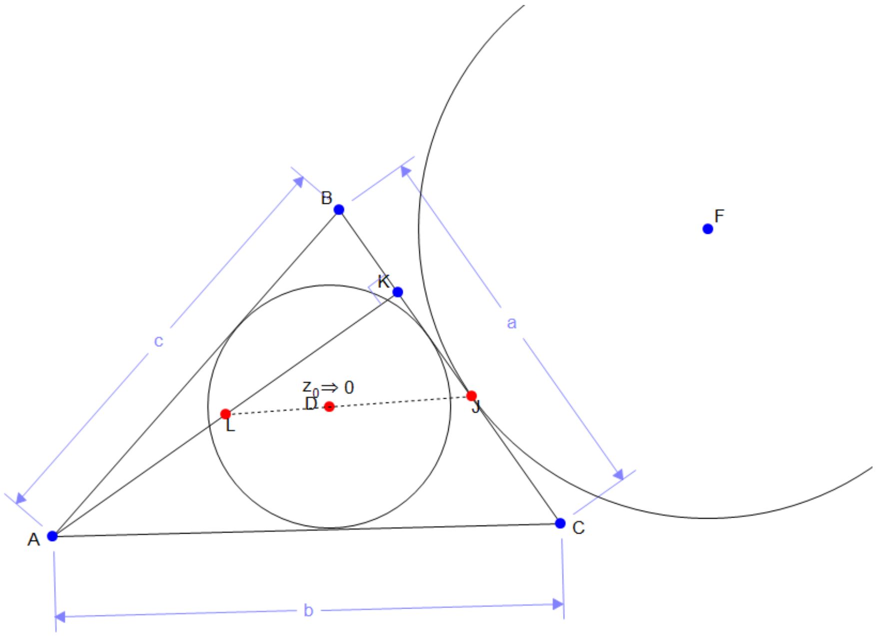
The midpoint of a side of a triangle, the foot of the altitude on this side, and the projections of the ends of this side upon the internal bisector of the opposite angle are four cyclic points.



We put the circle through E, F, G and check that DH is the same as the radius.

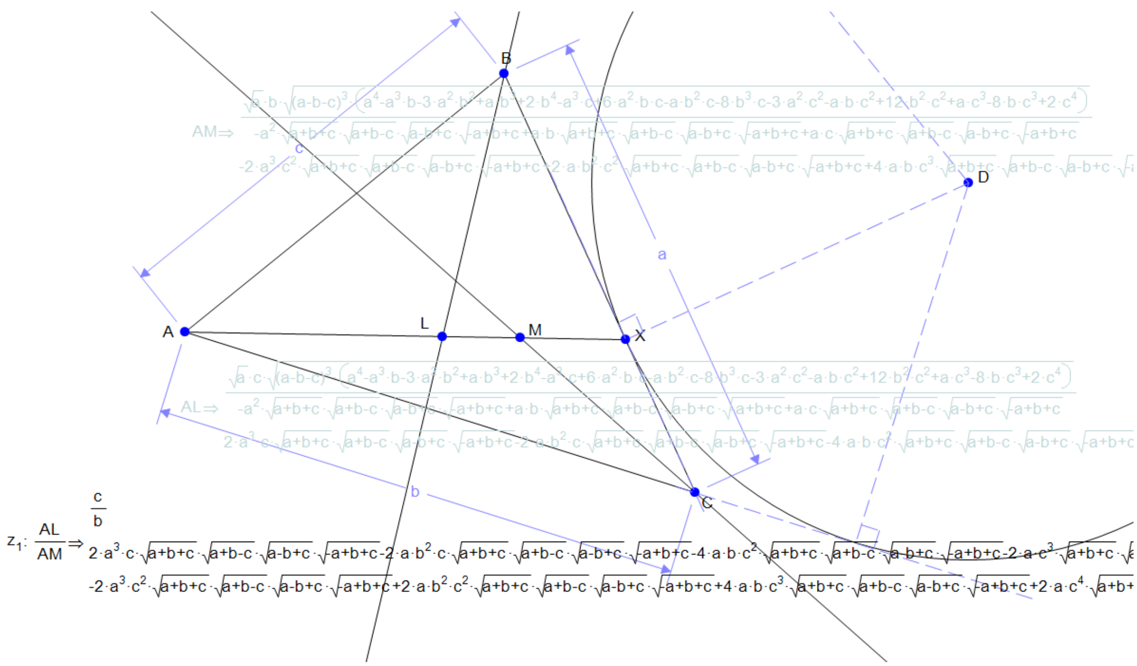
6.160

Show that the midpoint of an altitude of a triangle, the point of contact of the corresponding side with the excircle relative to that side and the incenter of the triangle are collinear



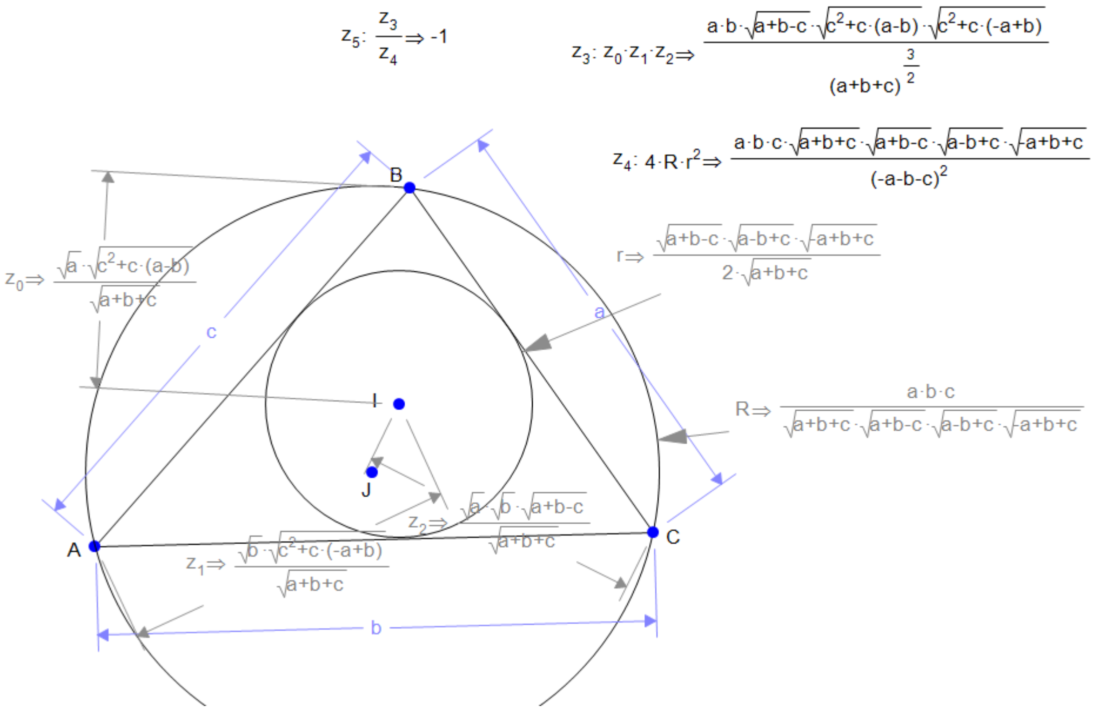
6.161

The internal bisectors of the angles B, C of the triangle ABC meet the line AX, joining A to the point of contact of BC with the excircle relative to this side in the points L, M. Prove that $\frac{AL}{AM} = \frac{AB}{AC}$.



6.162

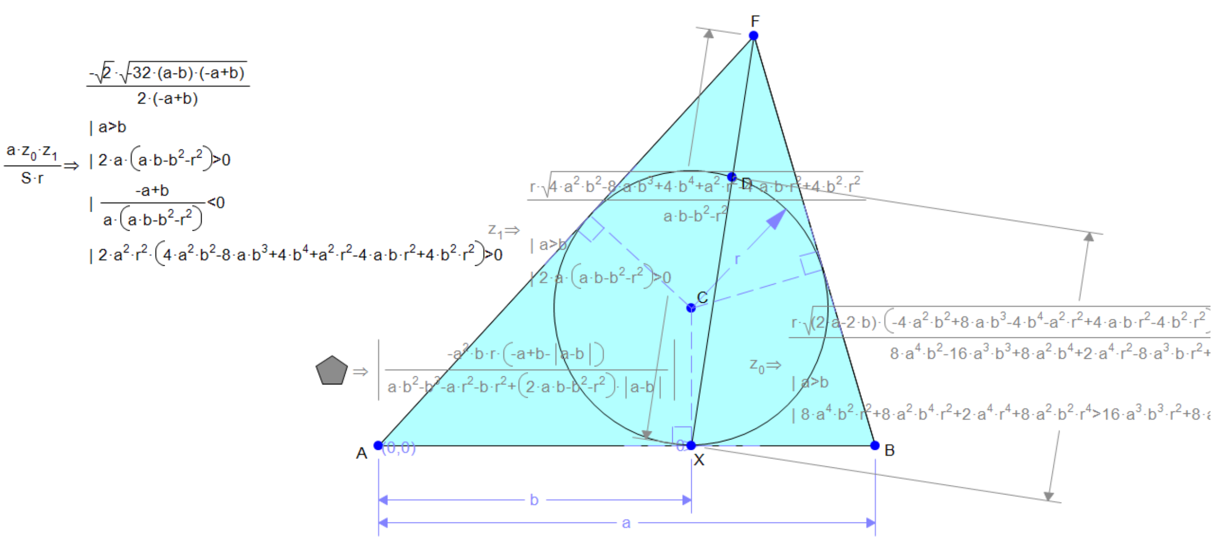
Show that the product of the distances of the incenter of a triangle from the three vertices of the triangle is equal to $4Rr^2$



We have the problem that the ratio comes out as -1. This is due to expressions not getting sensible witness values and needs fixed!

6.163

If the line AX joining the vertex of a triangle ABC with the point of contact X of the side BC with the incircle meets the circle again in X1, show that AX.AX1.BC = 4rS where r,S are the incircle radius and area of ABC.



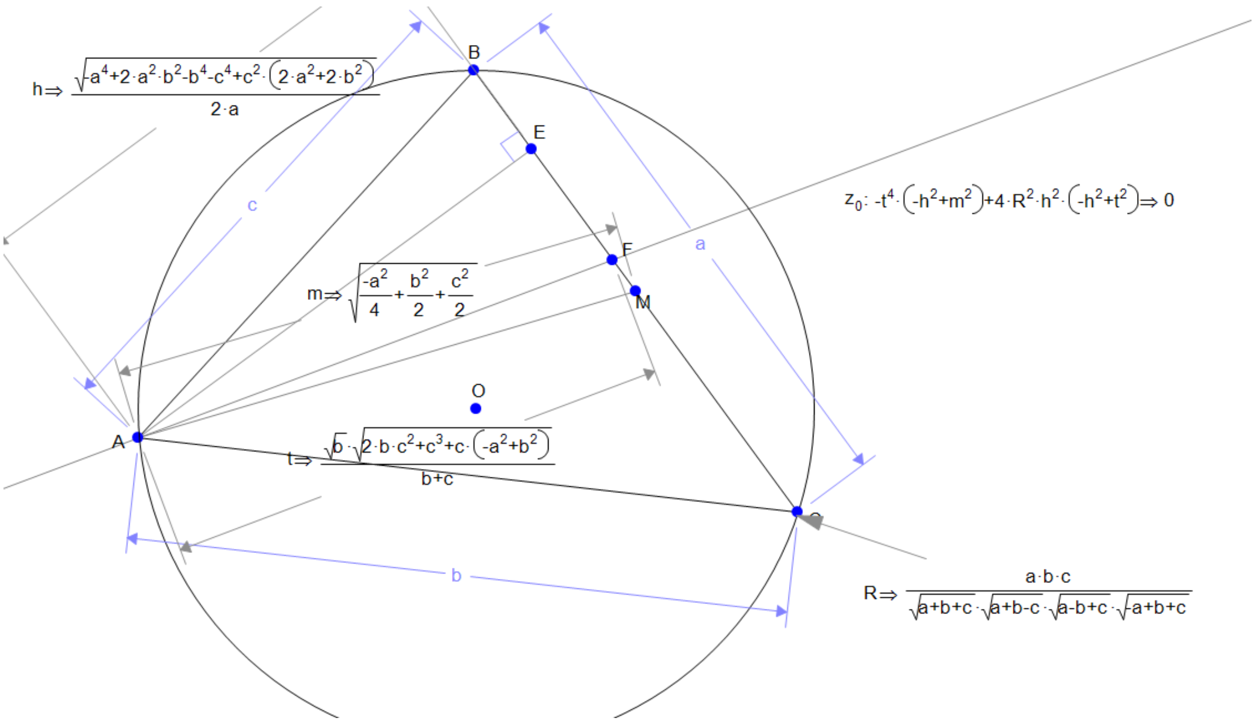
Defining this problem in the usual way, we did not get a resolution. So we resorted to defining it the way the book does, where the points A,B, I are independent.

Note however, that we are stumped in our simplification by the system not resolving (a-b)*(b-a) into

–(b-a)^2

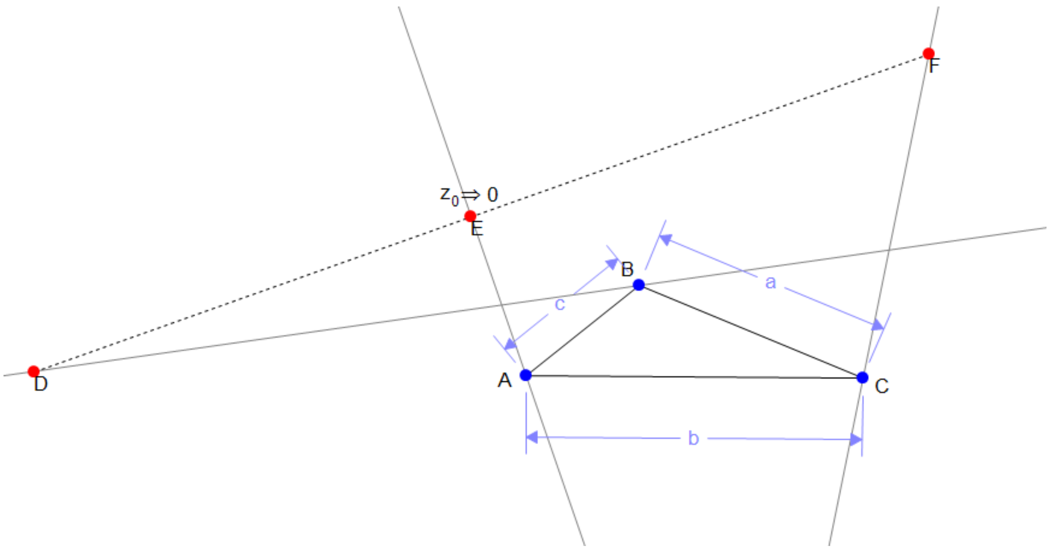
6.164

If h, m and t are the altitude, the median and the internal bisector issued from the same vertex of a triangle whose circumradius is R. Show that $4R^2h^2(t^2-h^2) = t^4(m^2-h^2)$



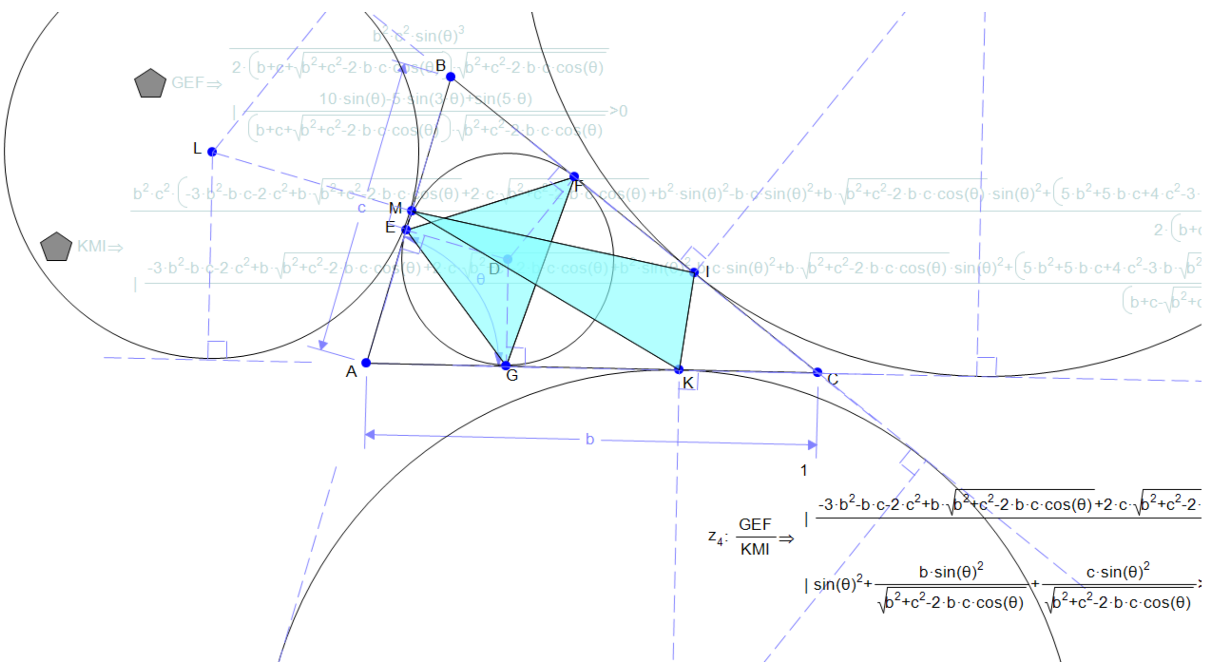
6.165

The external bisectors of the angles of a triangle meet the opposite sides in 3 collinear points



6.166

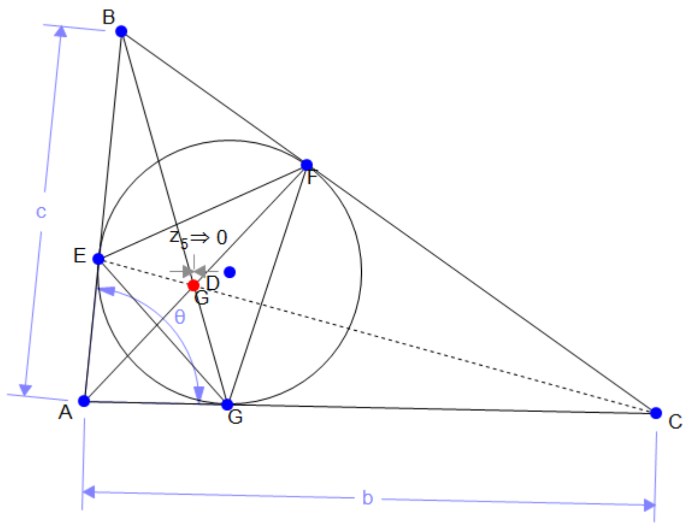
Prove that the triangle formed by the points of contact of the sides of a given triangle with the excircles corresponding to these sides has the same area with the triangle formed by the points of contact of the sides of the triangle with the inscribed circle.



This one had problems simplifying when we used the usual a,b,c definition of the triangle, but worked out when we constrained it with 2 sides and an angle.

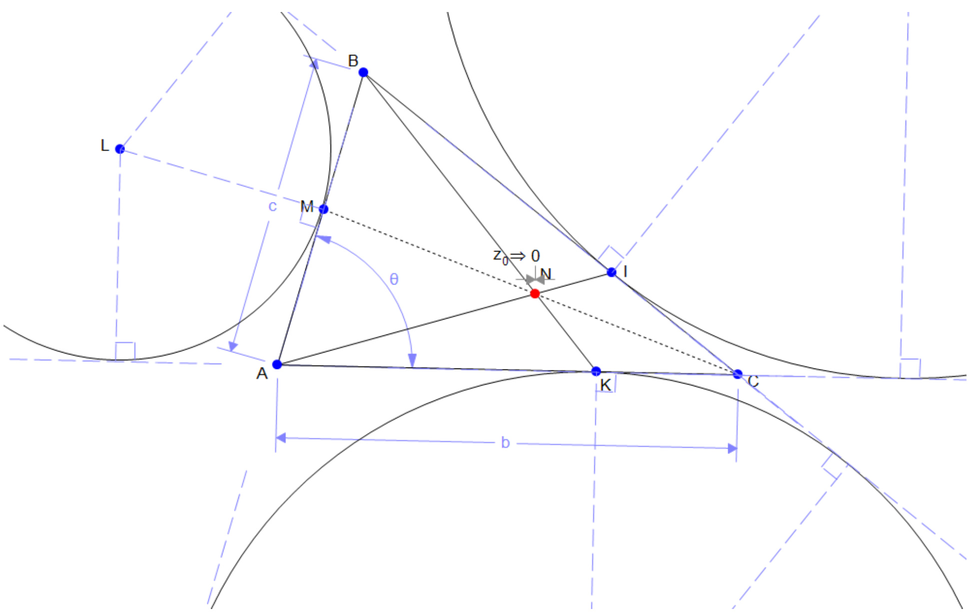
6.167 Gergonne Point

Show that the lines joining the vertices of a triangle with the points of contact of the opposite sides with the inscribed circle are concurrent (The Gergonne Point)



6.168 The Nagel Point

The lines joining the vertices of a triangle to the points of contact of the opposite sides with the excircles relative to those sides are concurrent (the Nagel Point)



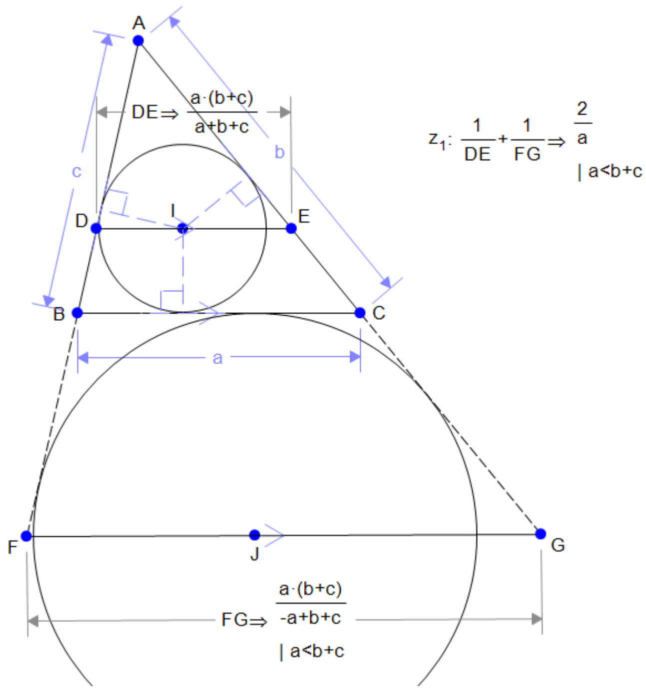
6.169

Show that the line joining the incenter of the triangle ABC to the midpoint of the segment joining A to the Nagel Point of ABC is bisected by the median issued from A.

This one did not work out. Simplification not sufficient. Nor was it good enough in Maple.

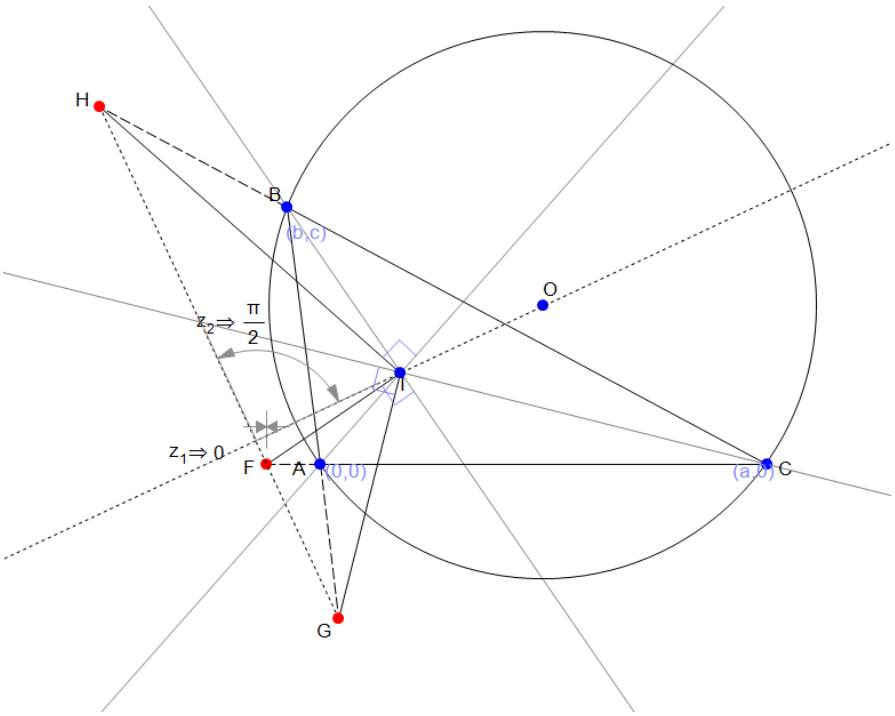
6.170

The sides AB, AC intercept the segments DE, FG on the parallels to the side BC through the tritangent centers I and J. Show that $\frac{2}{BC} = \frac{1}{DE} + \frac{1}{FG}$



6.171

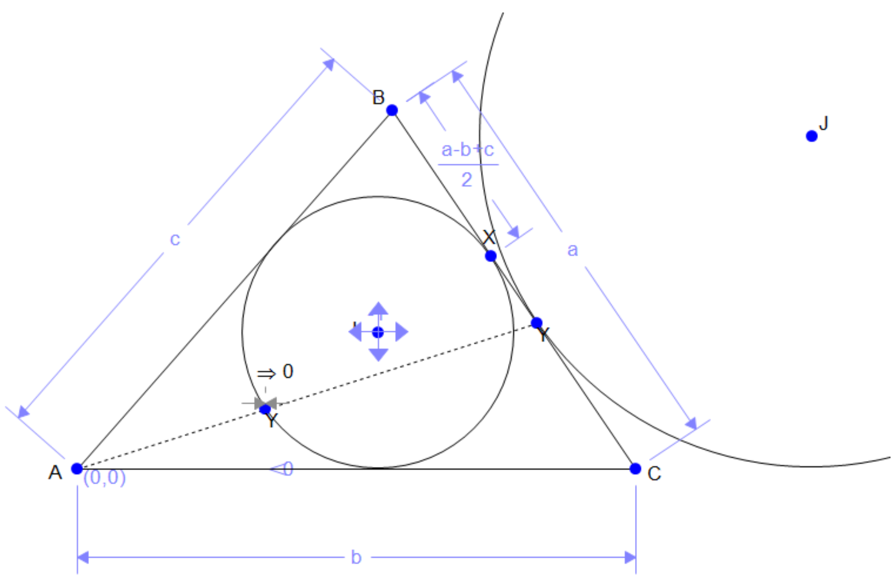
Show that the perpendiculars to the internal bisectors of a triangle at the incenter meet the respective sides in three points lying on a line perpendicular to the line joining the incenter to the circumcenter of the triangle



This one needed constrained by coordinates.

6.172

The side BC of the triangle ABC touches the incircle (I) in X and the excircle (J) relative to BC in Y. Show that the line AY passes through the diametric opposite Z of X in (I).

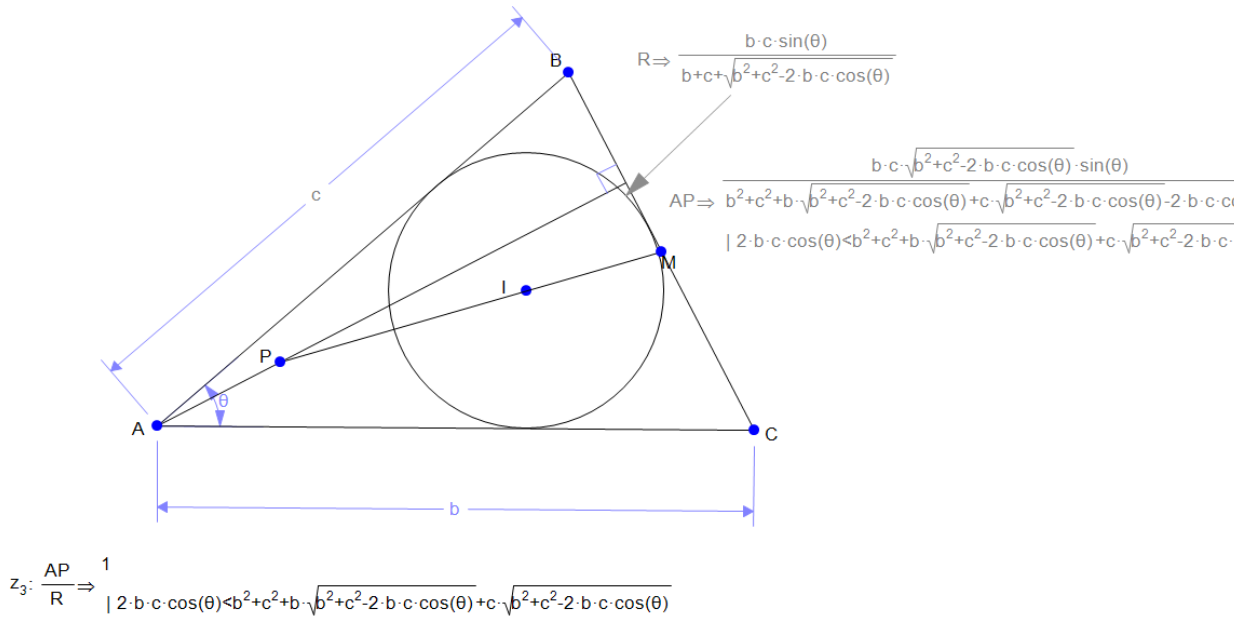


This one was not working, so we had to get creative… using a previous result that the distance BX is $\frac{a-b+c}{2}$

Could this be built in?

6.173

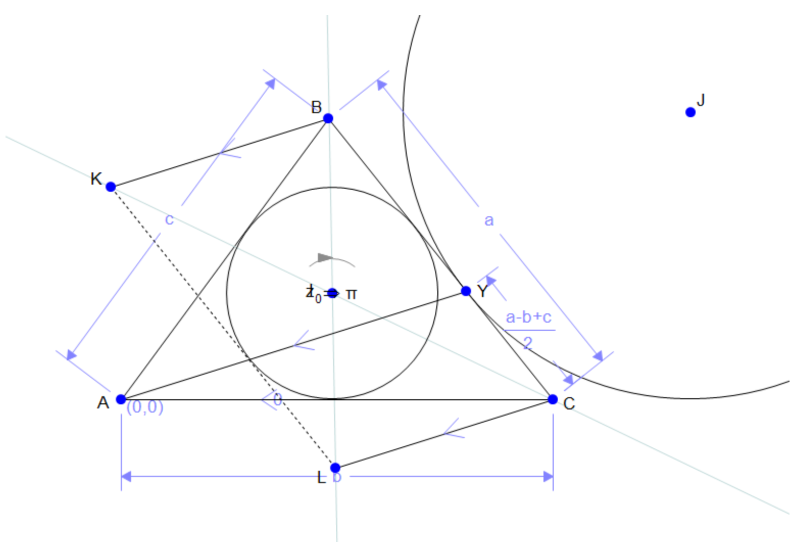
With the notation of 6.172, and with M the midpoint of BC show that if the line MI meets the altitude AD of ABC in P then AP is equal to the inradius of ABC.



We needed to constrain with an angle to get this one to simplify. Can we somehow force factoring of Heron's formula?

6.174

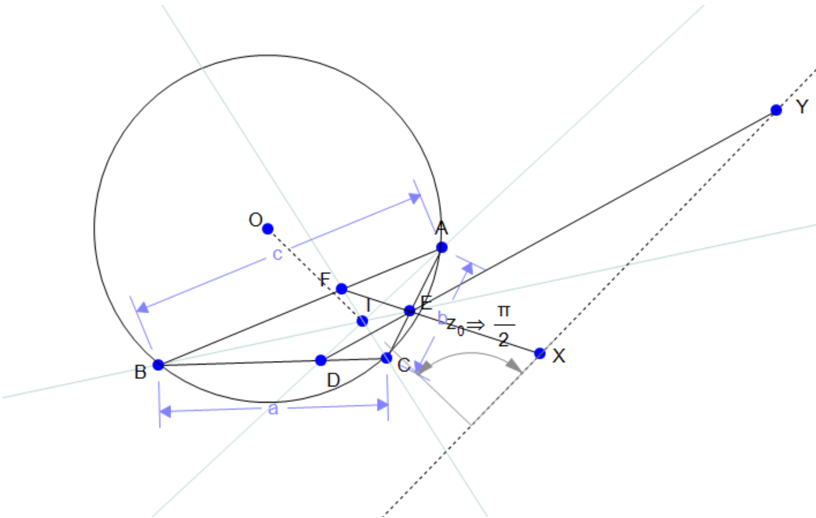
With the notations of 6.172, if the parallels to AY through B and C meet the bisectors CI and BI in L, M show that LM is parallel to BC.



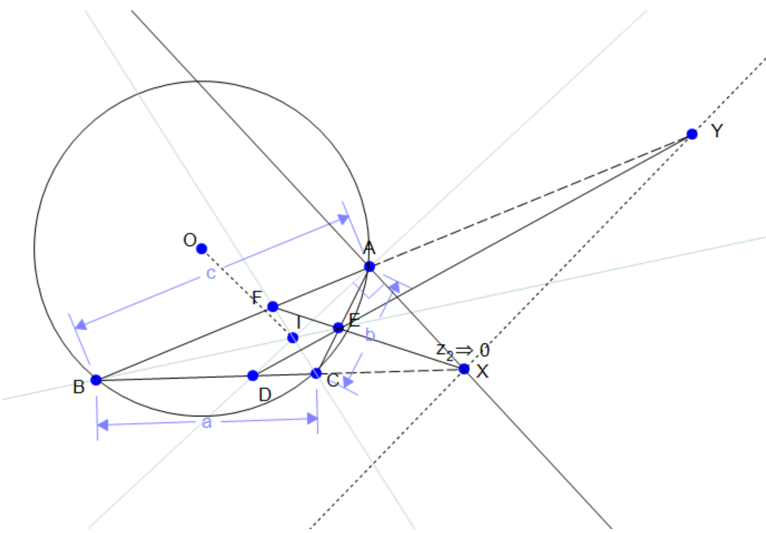
Again we used the hack of 6.172, remembering the distance of CY.

6.175

Show that the trilinear polar of the incenter of a triangle passes through the feet of the external bisectors and this line is perpendicular to the line joining the incenter and circumcenter



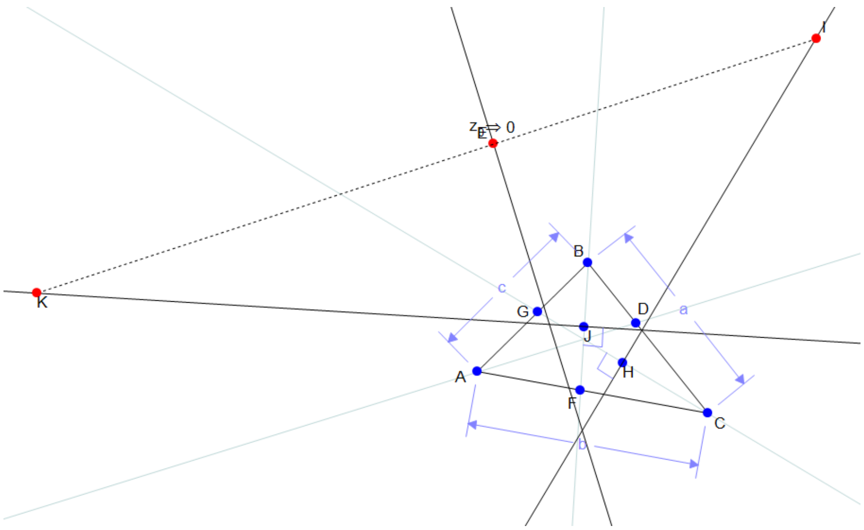
The diagram shows the second part of the theorem.



Here is the first part: we show that point X lies on the external bisector at A.

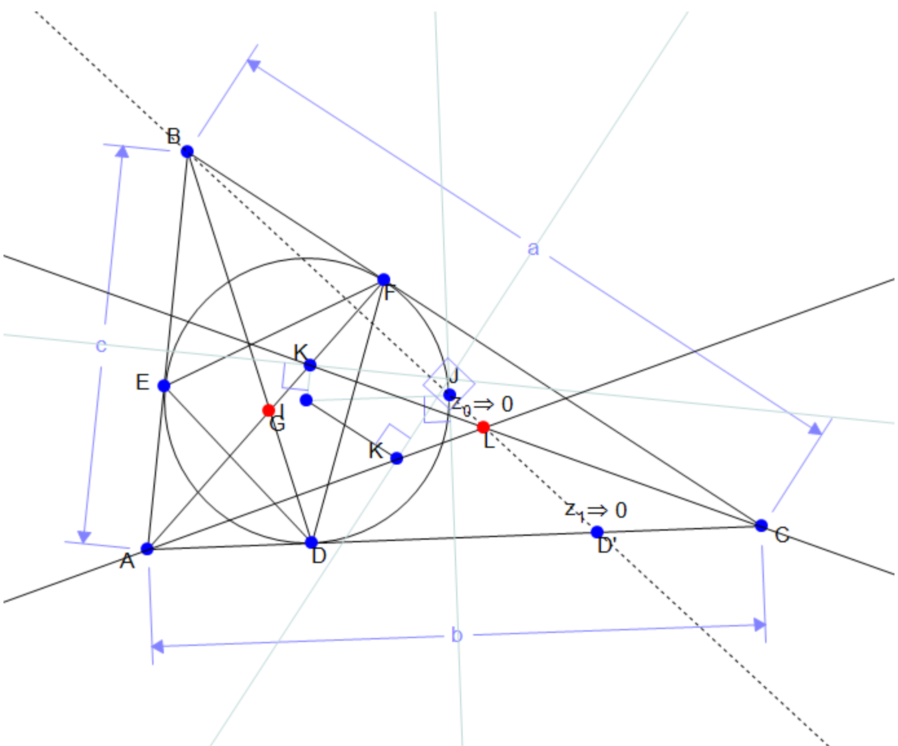
6.176

Show that the mediators of the internal bisectors of the angles of a triangle meet the respective sides of the triangle in 3 collinear points



6.177

Show that the lines joining the vertices of a triangle to the projections of the incenter upon the mediators of the respectively opposite sides meet in a point – the isotomic conjugate of the Gergonne Point of the triangle.

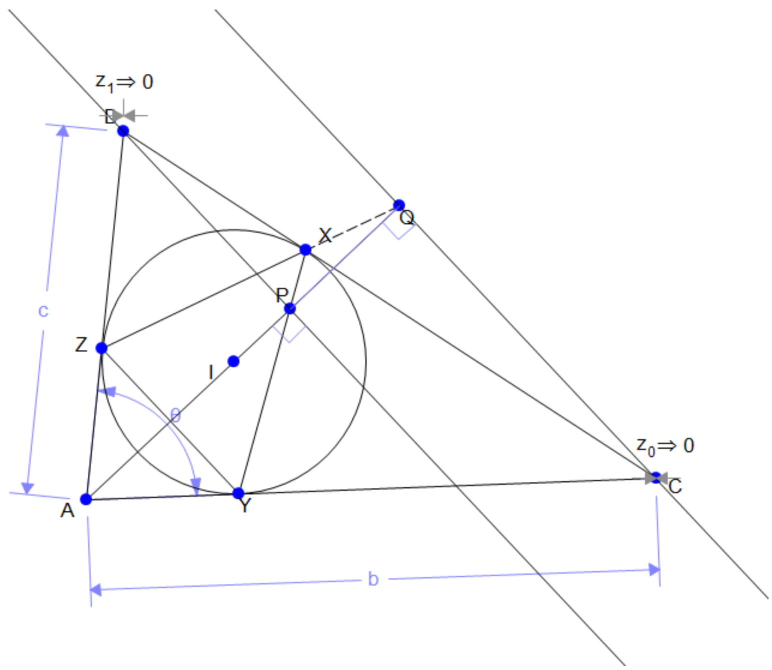


L lies on BJ. So does D' the reflection of D in the perpendicular bisector of AC. Hence and by symmetric arguments on the other sides, L is the isotomic conjugate of the Gergonne point G.

6.178

Show that the line AI meets the sides XY, XZ in two points P,Q inverse with respect to the incircle (I) = XYZ. And the perpendiculars to AI at P, Q pass through the vertices B, C of the given triangle ABC.

Here is the proof of the second part.

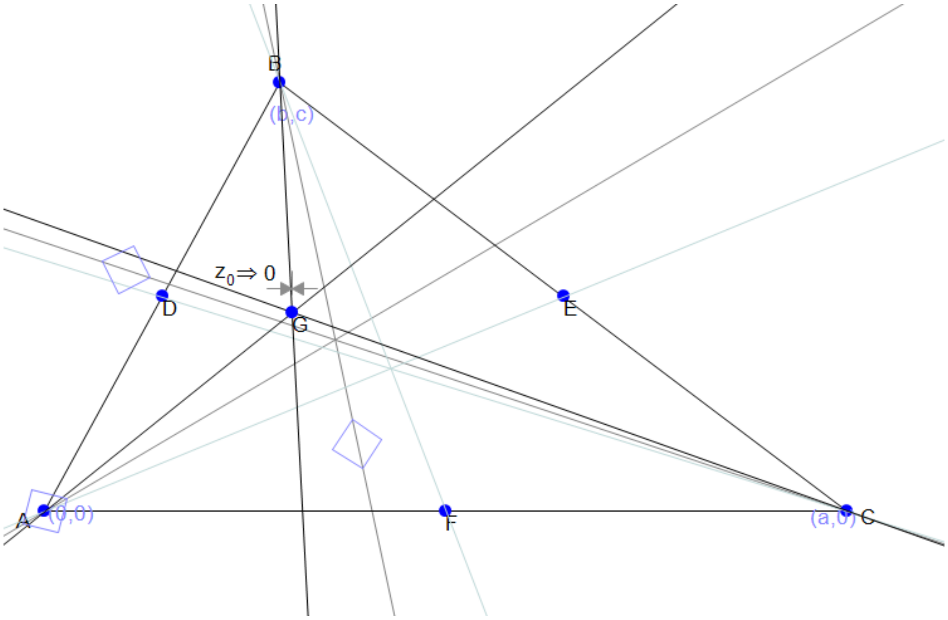


Definition

The symmetric of a median of a triangle with respect to the internal bisector issued from the same vertex is the symmedian of the triangle.

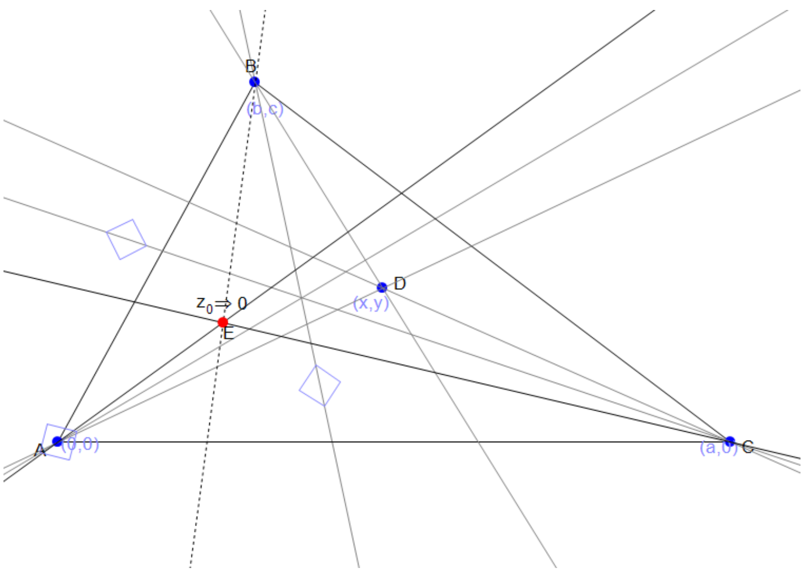
6.179

The three symmedians of a triangle are concurrent (The Lemoine Point, or the Symmedian Point).



6.180

The three symmetrics of the three lines joining a point and the three vertices of a triangle with respect to the internal bisectors issued from the same vertices are concurrent (the isogonal conjugate point).



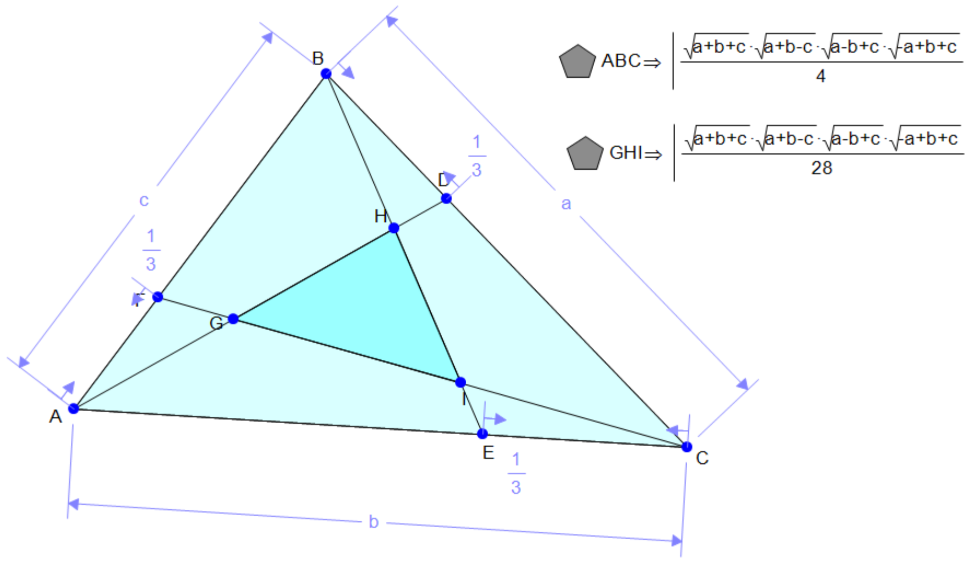
2.3.7 Intercept Triangles

Let L, M, N be three points on the sides BC, CA, AB of triangle ABC. Then triangle LMN and the triangle determined by lines AL, BM and CN are called the intercept triangles of triangle ABC for points L, M, N.

6.181

Let D,E,F be points on the sides BC, CA, AB of a triangle ABC such that BD/BC = CE/CA = AF/AB = 1/3.

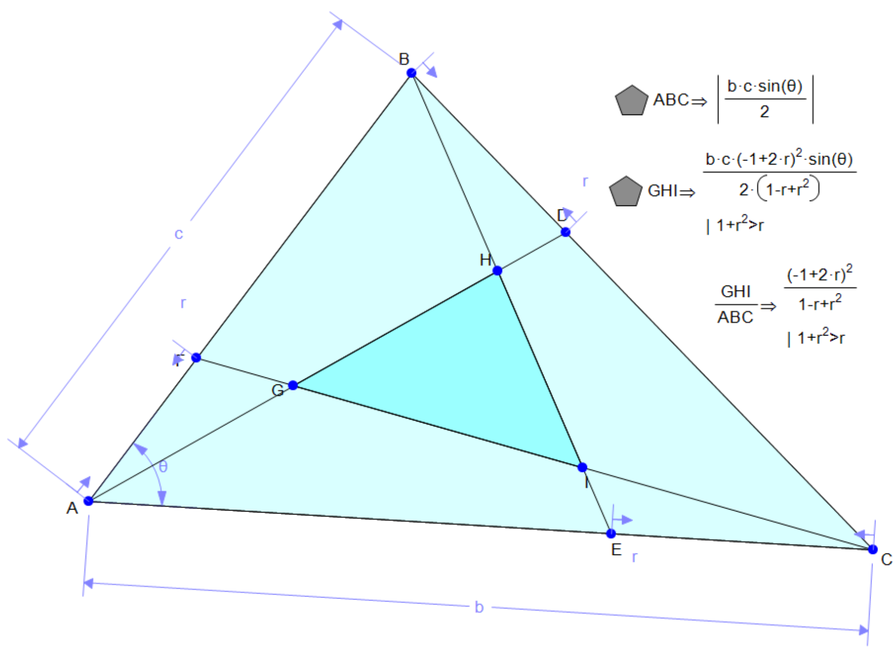
Show the area of the triangle determined by the lines AD, BE, CF is one seventh the area of triangle ABC.



6.182

Let D,E,F be points on the sides BC, CA, AB of a triangle ABC such that BD/BC = CE/CA = AF/AB = r.

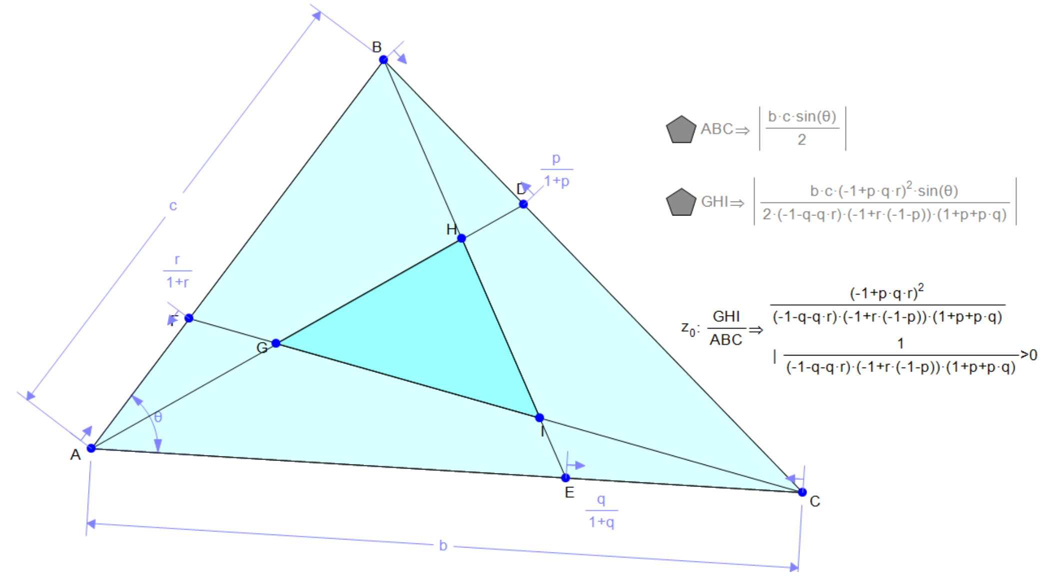
Show the ratio of area of the triangle determined by the lines AD, BE, CF to the area of triangle ABC is $\frac{(2\cdot r-1)^2}{r^2-r+1}$



We had to use the angle here to prevent the Heron's version of the area getting tangled up in the simplification.

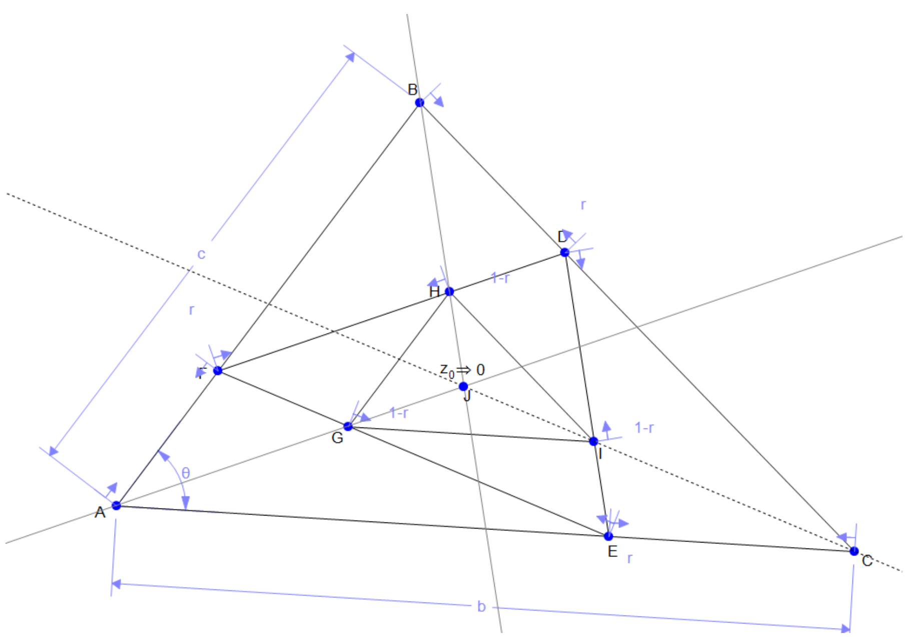
6.183

Using the above notation if $\frac{BD}{DC} = p, \frac{CE}{EA} = q, \frac{AF}{FB} = r$, then the ratio of areas is $\frac{(pqr-1)^2}{(qp+p+1)(rp+r+1)(rq+q+1)}$



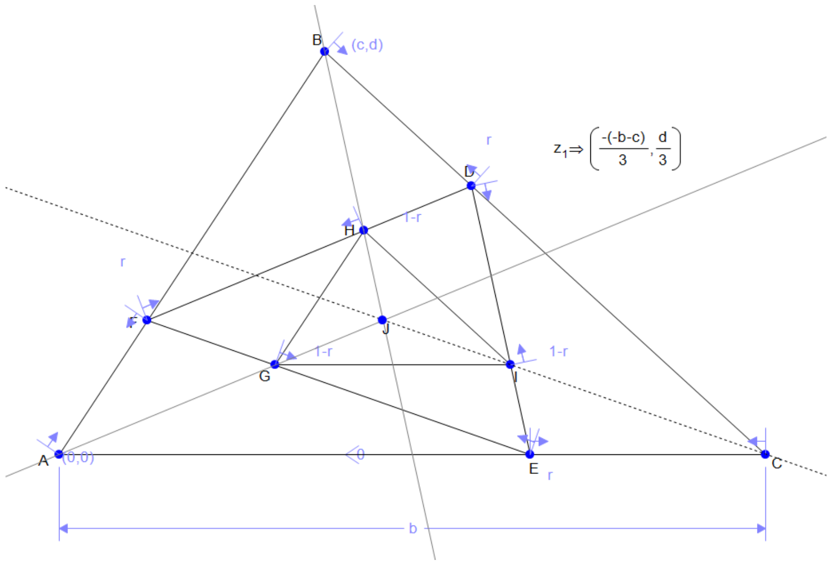
6.184

Let D,E,F be points proportion r along sides BC, CA and AB. Let G,H,I be points proportion (1-r) along sides DE, EF, FG. Show triangles ABC and GHI are homothetic



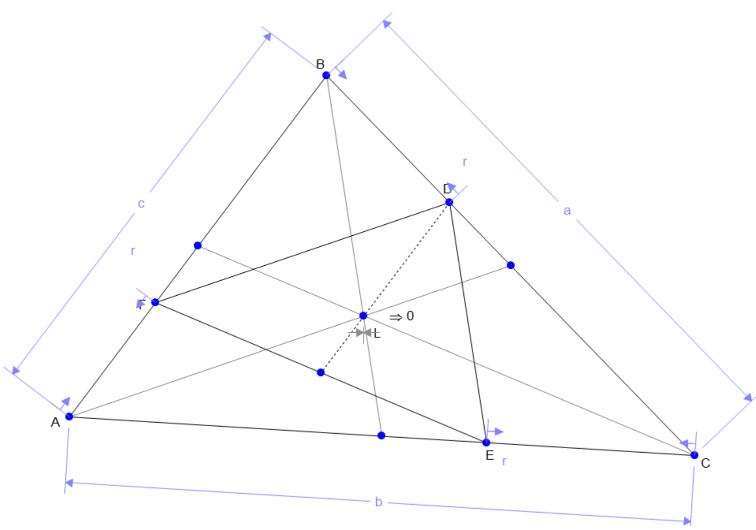
We show that the intersection of AG and BH lies on line CI.

In fact the center of the homothetic center is the centroid of ABC as is obvious from the coordinate picture



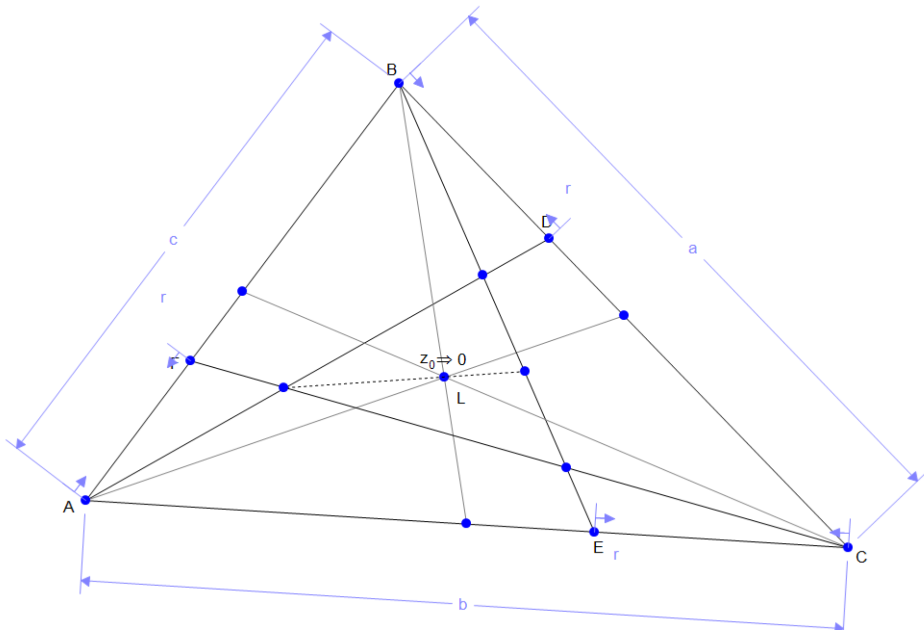
6.185

Let D,E,F be points an equal proportion along sides BC, CA, AB of triangle ABC. Show that the centroid of DEF is the centroid of ABC



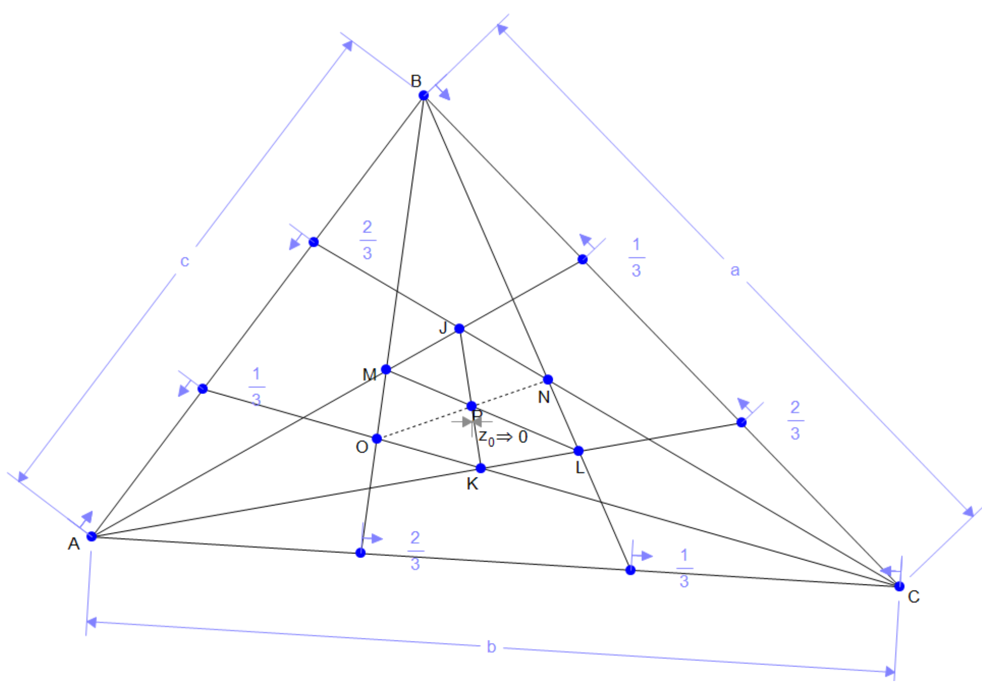
6.186

Let D,E,F be the same as in example 6.185. Show that the centroid of the triangle formed by AD, BE, CF coincides with the centroid of ABC.



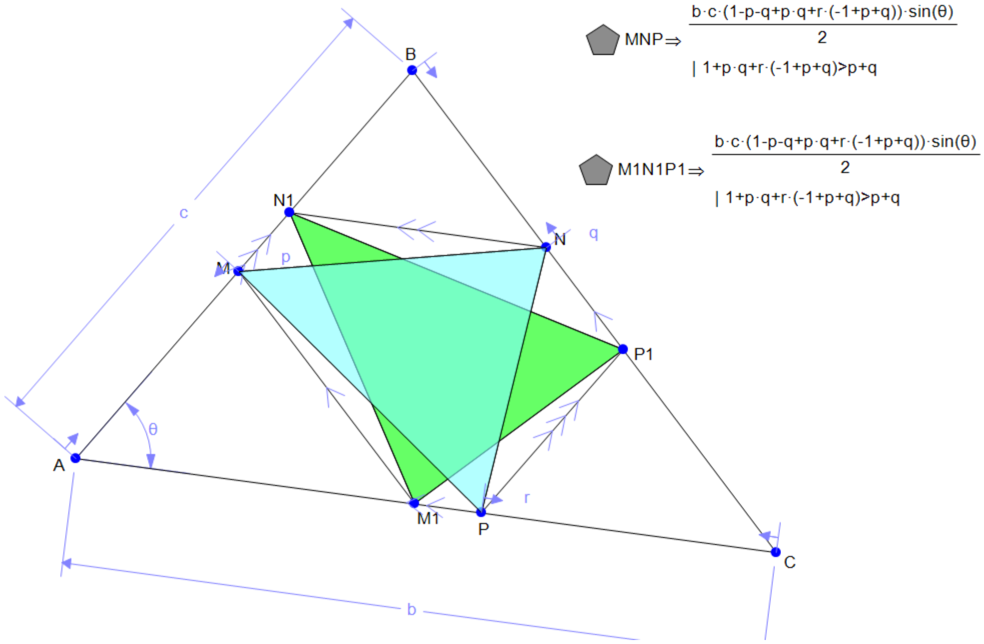
6.187

Through each of the vertices of the triangle ABC, we draw two lines dividing the opposite sides into three equal parts. These six lines determine a hexagon. Prove that the diagonals joining opposite sides of the hexagon meet in a point.



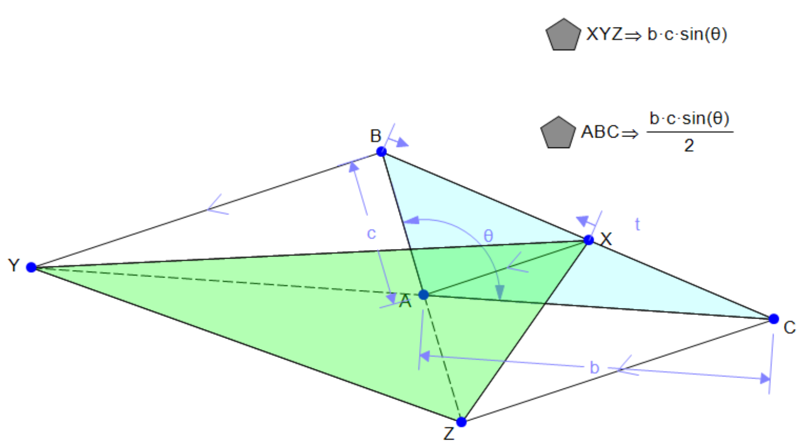
6.188

Let M,N,P be points on sides AB, BC, AC of a triangle ABC. Show that if M1, N1, P1 are points on sides AC, BA and BC such that MM1 is parallel to BC, NN1 is parallel to CA and PP1 is parallel to AB, then triangles MNP and M1N1P1 have the same areas.



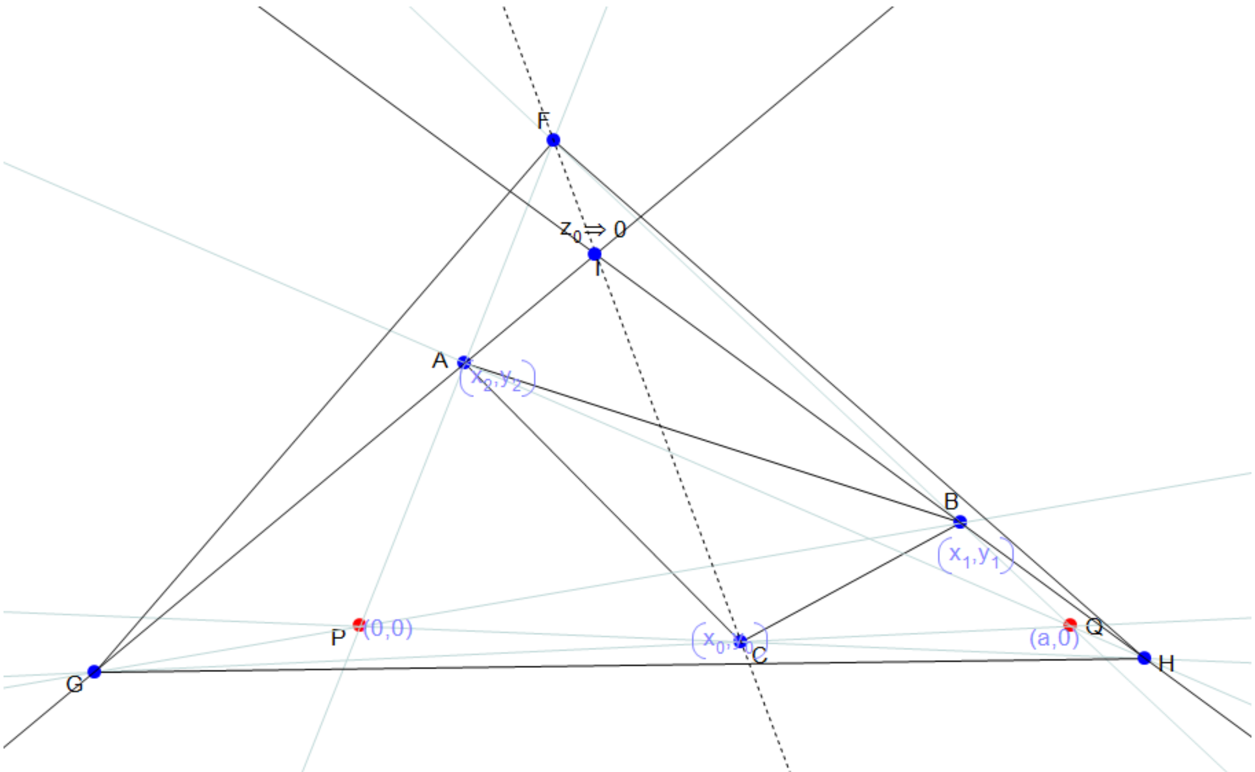
6.189

Three parallel lines drawn through the vertices of a triangle ABC meet the respectively opposite sides in the points X, Y, Z. Show that area XYZ is twice area ABC.



6.190

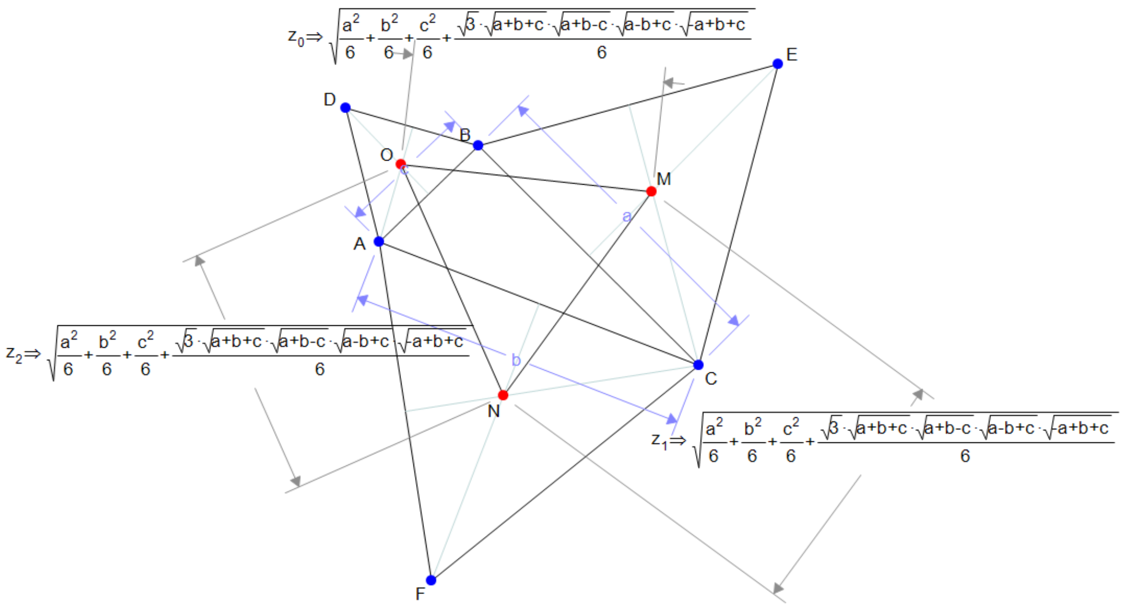
Two doubly perspective triangles are in fact triply perspective



2.3.8 Equilateral triangles

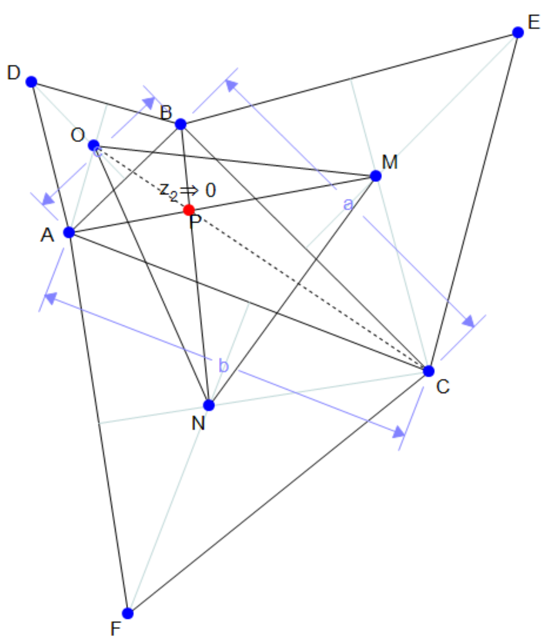
6.191 The Napoleon Triangle

If equilateral triangles are erected externally (or internally) on the sides of any triangle, their centers form an equilateral triangle.



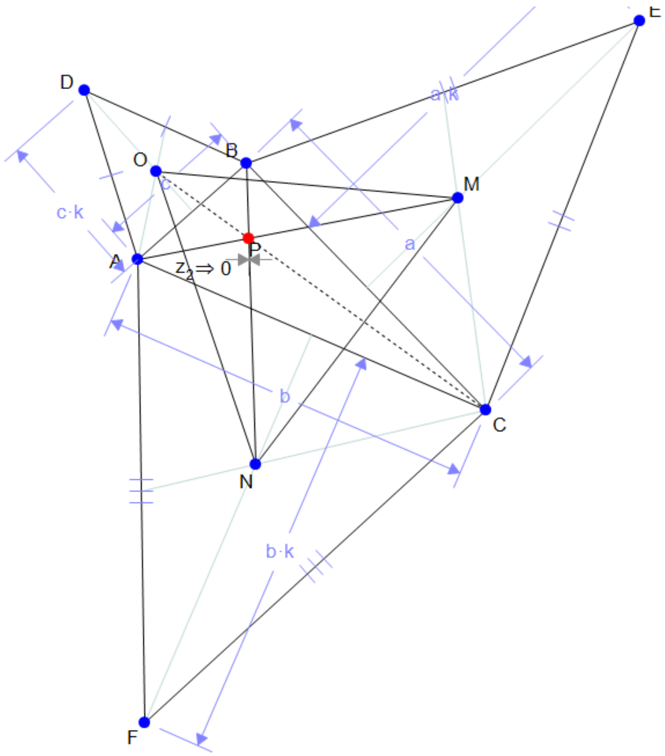
6.192

Continuing the above example, show that AM, BN and CO are concurrent



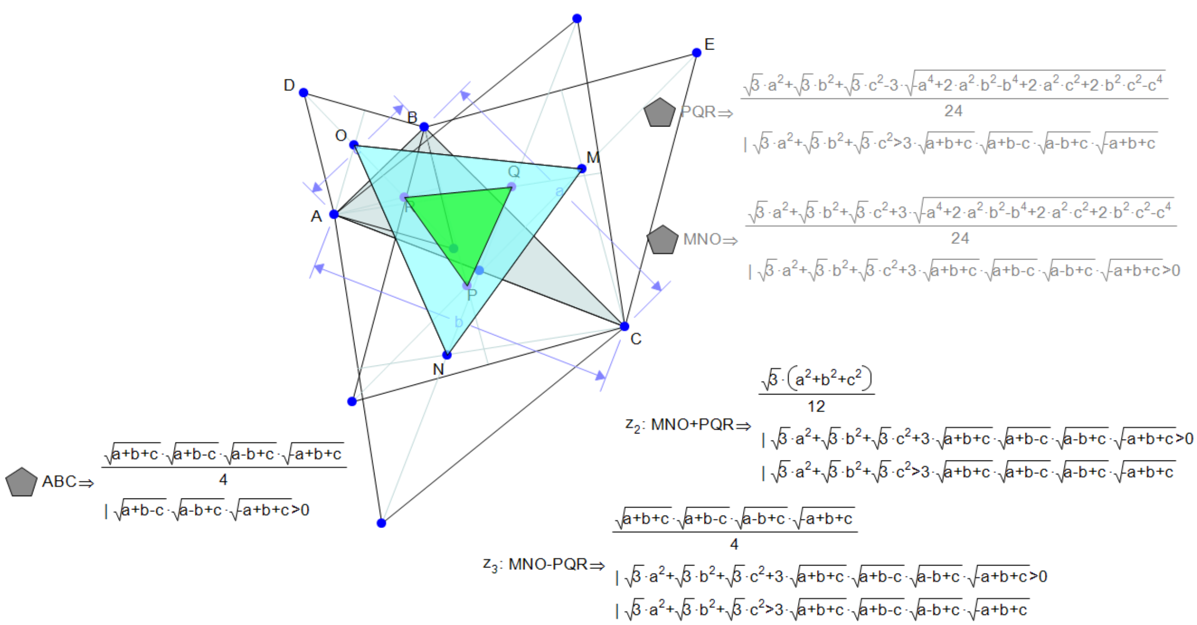
6.193

The above example is true for similar isosceles triangles.



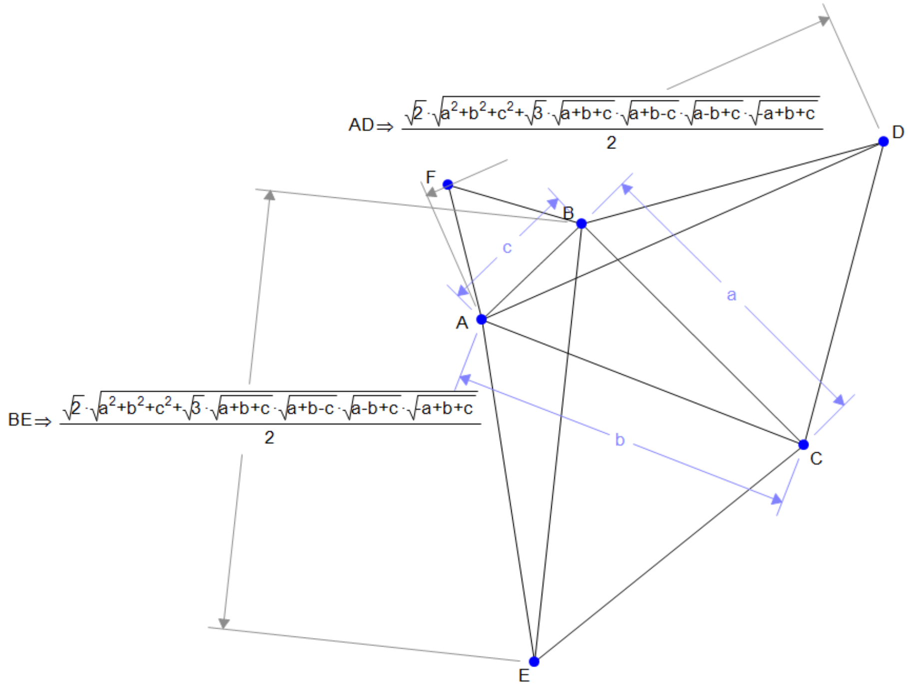
6.194

Let the centers of the equilateral triangles erected externally be M,N,O, and internally P,Q,R. The area of ABC is the difference of the areas of MNO and PQR



6.195

Let equilaterals BCD, ABF and ACE be erected externally on the sides of triangle ABC. Show that AD=CF=BE

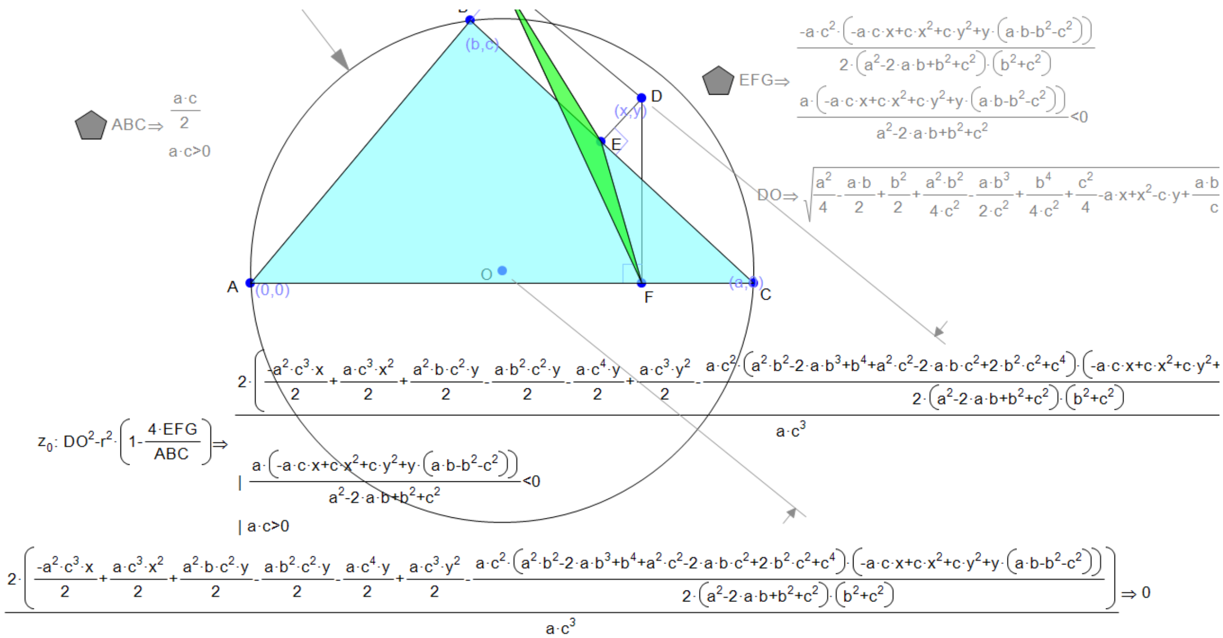


2.3.9 Pedal Triangles

From a point P three perpendicular lines are drawn to the sides of a triangle. The triangle whose vertices are the feet of these perpendiculars is called the pedal triangle of point P with respect to the given triangle

6.196

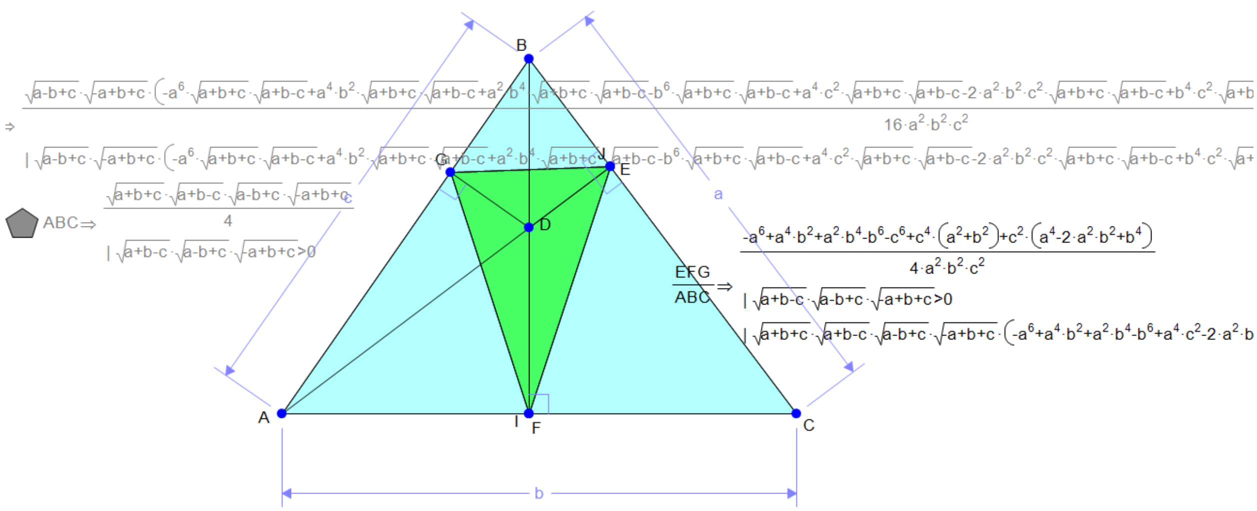
The orthogonal projections of D on BC, AC and AB are E,F,G respectively. Let O be the circumcenter of the triangle. Show that $DO^2 = AO^2\left(1 - \frac{4EFG}{ABC}\right)$



Used coordinates on this one, but do not get a simplification. However, if I copy and paste the result into a new Expression, I do get the simplification. This should be investigated.

6.197

Let K be the area of the pedal triangle of the orthocenter of ABC. What is the ratio of K to ABC?



Maple gave us factors:

```
> factor(1/4*(-a^6+b^2*a^4+b^4*a^2-b^6-c^6+(a^2+b^2)*c^4+(a^4-
2*b^2*a^2+b^4)*c^2)/c^2/b^2/a^2);
```

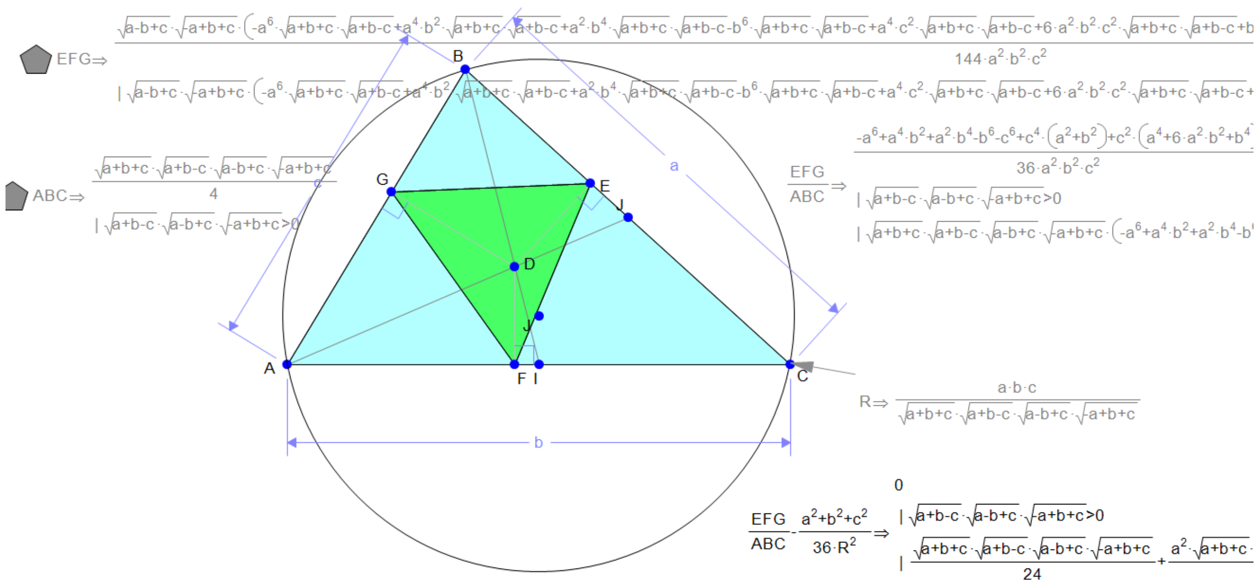
$$-\frac{(b^2+a^2-c^2)(-b^2+a^2+c^2)(-b^2+a^2-c^2)}{4\,c^2\,b^2\,a^2}$$

Which are in the form of the book.

6.198

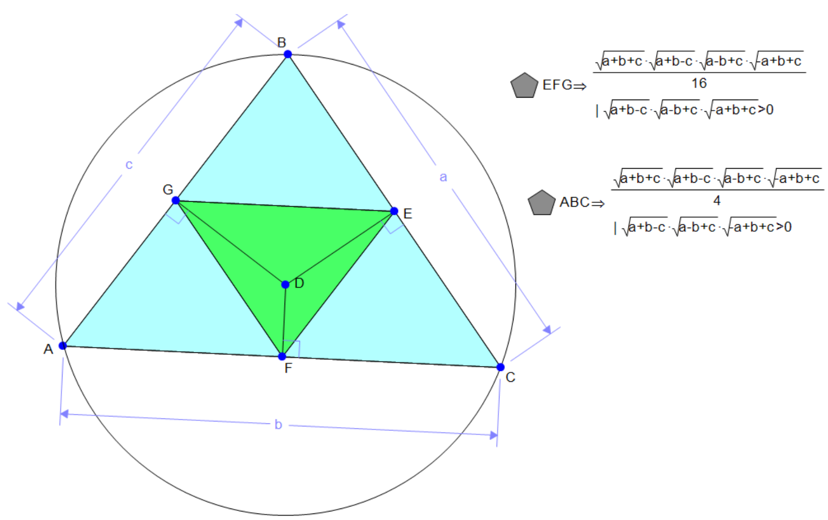
Let K be the area of the pedal triangle of the centroid of ABC and R the circumradius of ABC.

Show that $\frac{K}{ABC}=\frac{AB^2+BC^2+AC^2}{36R^2}$



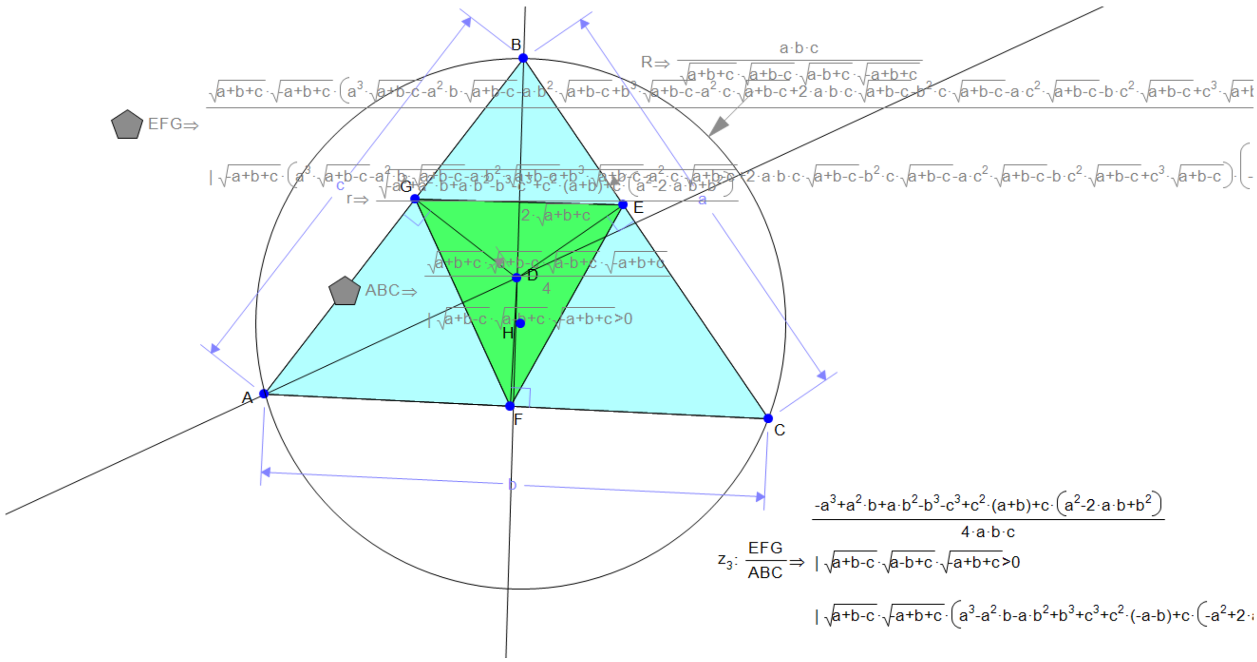
6.199

Let K be the area of the pedal triangle of the circumcenter of ABC. Show that the area of ABC is 4K.



6.200

Let K be the area of the pedal triangle of the incenter of ABC. Show that $\frac{K}{ABC} = \frac{r}{2R}$



Maple factored this into a workable form:

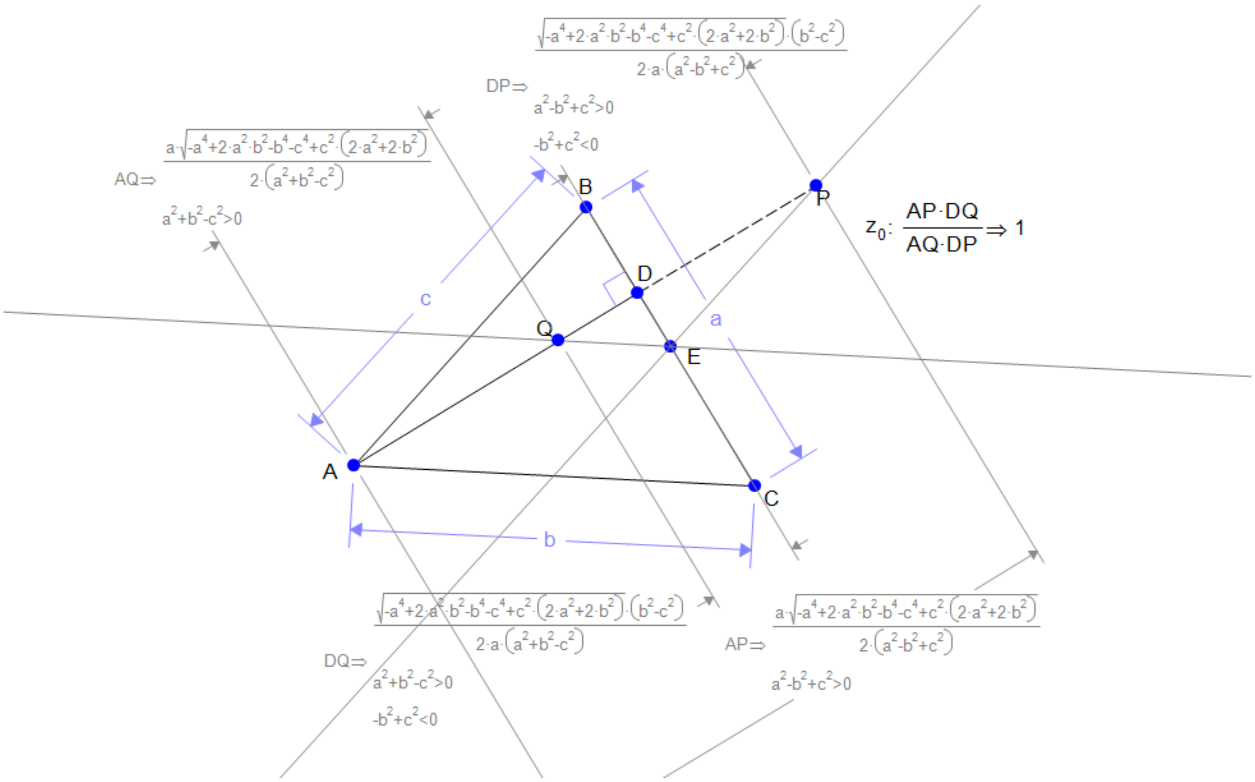
```
> factor(1/4*(-a^3+b*a^2+b^2*a-b^3-c^3+(a+b)*c^2+(a^2-
2*b*a+b^2)*c)/c/b/a);
```

$$-\frac{(a+b-c)(a-b+c)(a-b-c)}{4\,c\,b\,a}$$

3.10 Miscellaneous

6.201

AD and AE are the altitude and the median of the triangle ABC; the parallels through E to AB, AC meet AD in P, Q; Show that ADPQ are a harmonic range (|AP||DQ| = |AQ||DP|)



6.202

If E,F,G are the midpoints of the sides of the triangle ABC, and G and H are the intersections of AD and ED with CF, show that CHGF form a harmonic range (|CH||FG|=|FC||GH|)

$$FG \Rightarrow \sqrt{\frac{a^2}{18}+\frac{b^2}{18}-\frac{c^2}{36}}$$

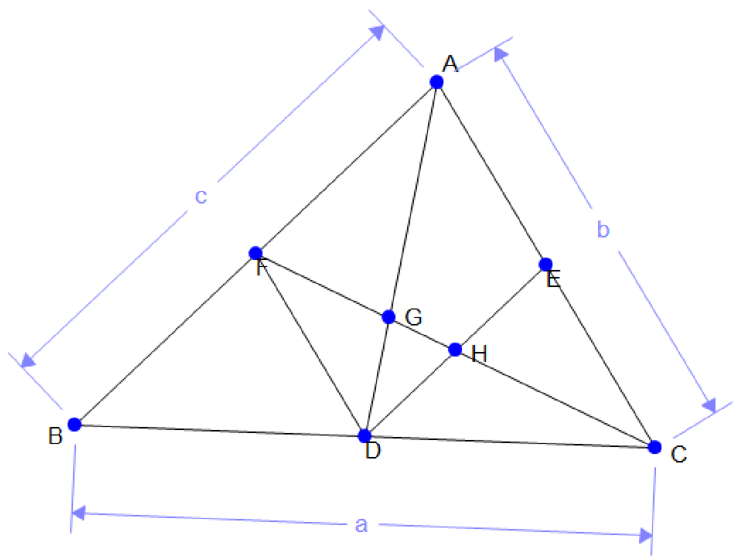
$$CH \Rightarrow \sqrt{\frac{a^2}{8}+\frac{b^2}{8}-\frac{c^2}{16}}$$

$$GH \Rightarrow \sqrt{\frac{a^2}{72}+\frac{b^2}{72}-\frac{c^2}{144}}$$

$$FC \Rightarrow \sqrt{\frac{a^2}{2}+\frac{b^2}{2}-\frac{c^2}{4}}$$

$$CH \cdot FG \Rightarrow \frac{a^2}{12}+\frac{b^2}{12}-\frac{c^2}{24}$$

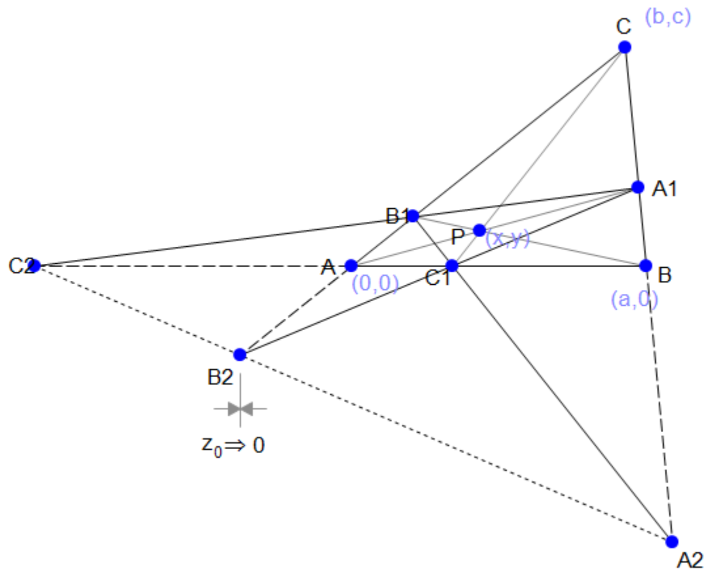
$$FC \cdot GH \Rightarrow \frac{a^2}{12}+\frac{b^2}{12}-\frac{c^2}{24}$$



6.203

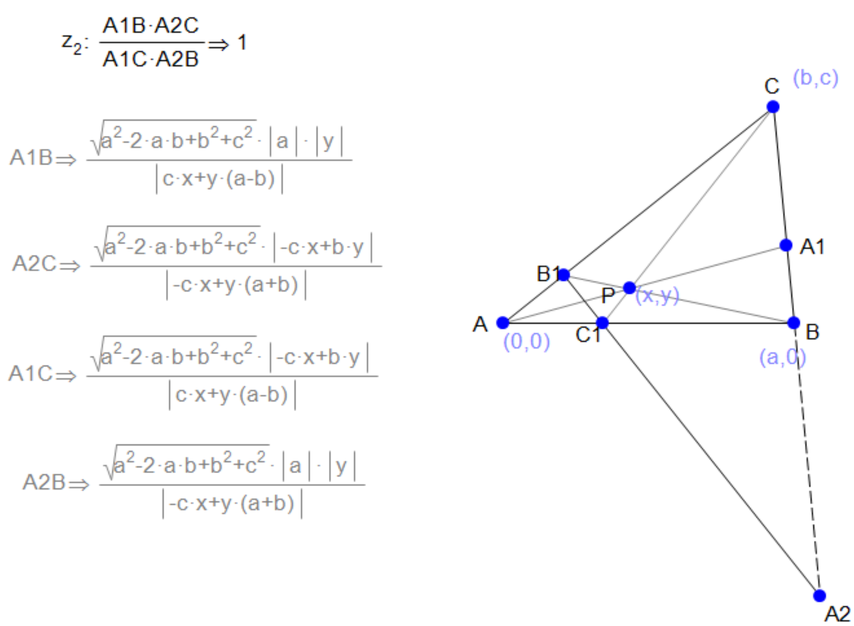
Let P be a point in the plane of the triangle ABC. Let A1 be the intersection of BC and AP, B1 be the intersection of BP with AC and C1 be the intersection of CP with AB. Further, let A2 be the intersection of BC with B1C1, B2 be the intersection of AC with A1C1 and C2 the intersection of AB with A1B1.

Show that A2, B2 and C2 are collinear.



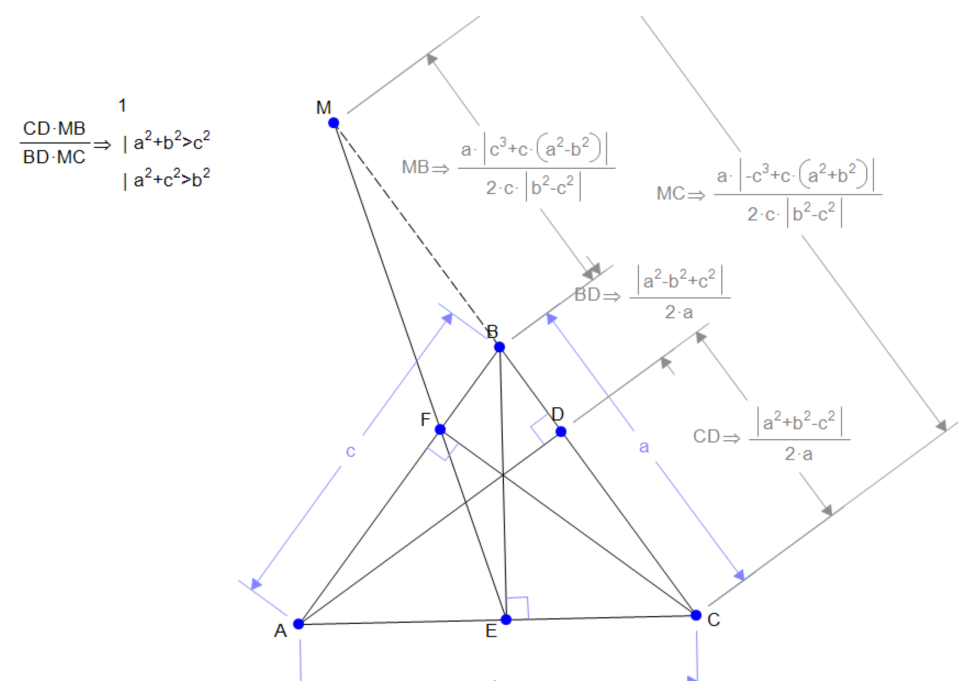
6.204

Let P be a point in the plane of the triangle ABC. Let A1 be the intersection of BC and AP, B1 be the intersection of BP with AC and C1 be the intersection of CP with AB. Further, let A2 be the intersection of BC with B1C1. Show that A1,A2,B,C form a harmonic sequence (|A1B||A2C|=|A1C||A2B|)



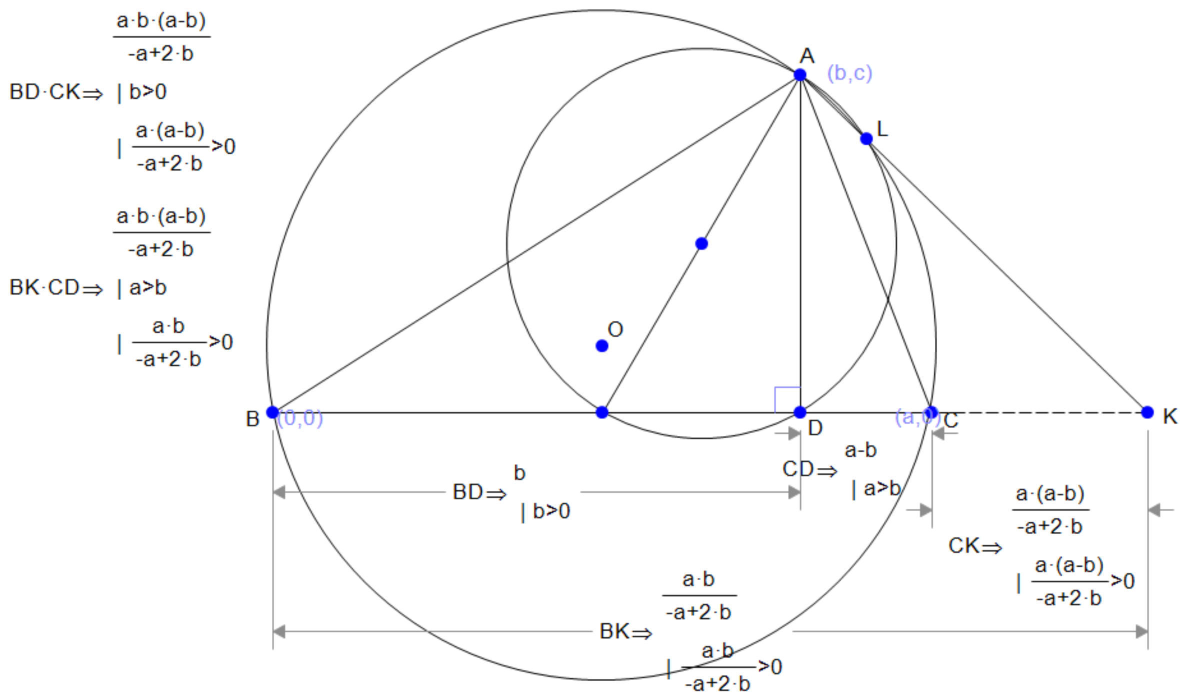
6.205

Let AD, BE, CF be the altitudes of triangle ABC. If EF meets BC in M, show that MBDC are a harmonic range.



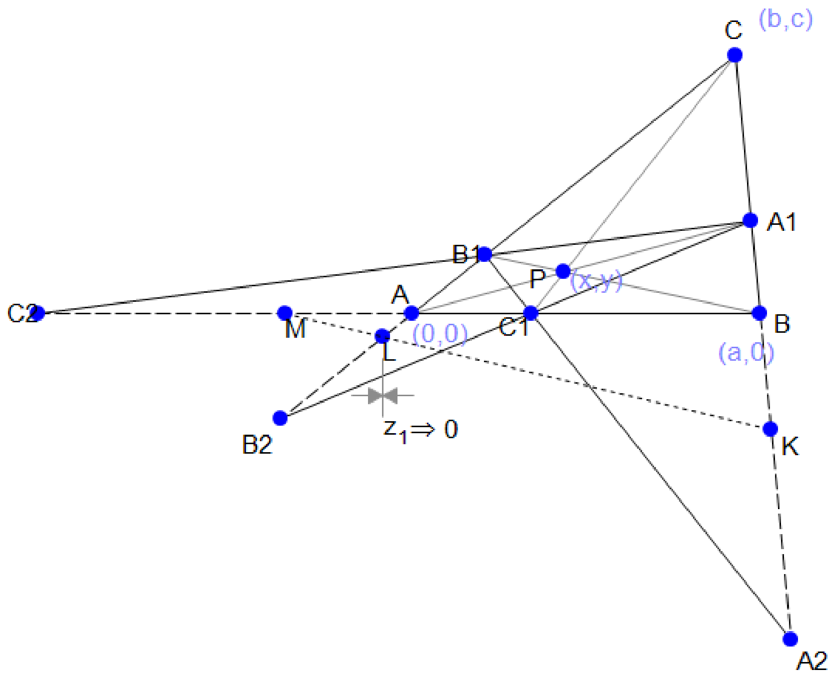
6.206

The circle having for diameter the median AA1 of the triangle ABC meets the circumcircle in L. Let AL meet BC in K. Show that KDBC is a harmonic range.



6.207

Using the diagram of 6.203, let K be the midpoint of A, A1 and L the midpoint of B, B1 and M the midpoint of C,C1. Show K, L, M are collinear.



References

[1] Chou, S. C., Gao, X. S., & Zhang, J. (1994). Machine proofs in geometry: Automated production of readable proofs for geometry theorems. World Scientific.